Cauchy-Schwarz and Means Inequalities for Selfadjoint Operator

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ABSTRACT

We determined that for bounded operators A and B on a Hilbert space there holds $\left|\left|\left| \, |A-B|^p \right|\right|\right| \le 2^{p-1} \left|\left|\left| A|A|^{p-1} - B|B|^{p-1} \right|\right|\right|$ for all $p \ge 2$, and if A and B are additionally self-adjoint operators, then

$$\left|\left|\left| \, |AX+XB|^p \right|\right|\right| \le 2^{p-1} |X|^{p-1} \left|\left|\left| \, |A|^{p-1} AX + XB|B|^{p-1} \right|\right|\right|$$

for all $p \ge 3$. We proved that if A and B are self-adjoint operators on $B(H)$ where $AX = XB$ then

$$\left|\left|\left| \, |AX+XB|^2 \right|\right|\right| \le 2 \left|\left|\left| \, |AX|^2 + |XB|^2 \right|\right|\right|.$$

Similarly the inequality $\left|\left|\left| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right|\right|\right| \le \frac{1}{2} ||| AX + XB |||$ is known. We proved that for self-adjoint normal contractions operators A and B then $\left|\left|\left| \left(I - |A^*|^2\right) X^2 \left(I - |B^*|^2\right) \right|\right|\right| \le \left|\left|\left| X - A^* X B^* \right|\right|\right|^2$ hold, for all $X \in B(H)$.

Keywords: Csauchy-schwarz - selfadjoint operator - Norm Inequalities

PRELIMINARIES

Recently, the following perturbation norm inequality has been established in [2].

Theorem 1.1: If A and B are self-adjoint operators in $B(H)$, then for all natural numbers n and every unitarily invariant norm $|||\cdot|||$,

$$|||(A-B)^{2n+1}||| \le 2^{2n} |||A^{2n+1} - B^{2n+1}|||. \qquad (1)$$

This completely resolves the problem raised by Koplienko and others (see [14] and [15]) for estimating $A-B$ when $A^n - B^n$ is given in a specified norm ideal.

Lemma 1.1: If self-adjoint A and B and an arbitrary X are in $B(H)$, then for all non-negative integers n and every unitarily invariant norm $|||\cdot|||$ there holds the following chain inequality

$$|||A^n(A-B)B^n||| \le A^{2-1}(A^3-B^3)B^{n-1}||| \le \cdots \le |||A^{2n+1} - B^{2n+1}|||.$$

This lemma itself was based on the following arithmetic-geometric mean inequality (see [16] and [6]), which we will also need in the sequel.

Theorem 1.2: For arbitrary A, B and X in $B(H)$, and every unitarily invariant norm $|||\cdot|||$,

$$2|||A^*XB||| \le |||AA^*X + XBB^*|||.$$

We will show that the inequality (1) also holds for all self adjoint derivations $AX - XB$ and for all real $n \geq 0$.

Let *B*(*H*) and ℓ_∞ denote respectively the space of all bounded and compact linear operators acting on a separable, infinite-dimensional, complex Hilbert space *H*. Following [10], for an arbitrary $A \in B(H)$, let $s_1(A) \geq s_2(A) \geq \cdots$ denote the singular values of *A*, i.e., the eigenvalues of $|A| = (A^*A)^{1/2}$ exceeding the essential norm $\|A\|_e = s_\infty(A) = \sup \sigma_{ess}(|A|)$, arranged in a non-increasing order, with their (necessarily finite) multiplicities counted. If necessary, this sequence can always be made infinite by adding $s_n(A) = s_\infty(A)$ for missing n. Note that for all bounded A we have $s_\infty(A) = \lim_{n\to\infty} s_n(A)$, while $A \in \ell_\infty$ if and only if $s_\infty(A) = 0$. For the extension of some standard singular value properties to bounded operators see [10] and [13].

Each symmetric gauge function Φ on sequences (see [10] for definition) gives rise to unitarily invariant norm on operator ideal ℓ_Φ contained in ℓ_∞ which is complete in the topology induced by the norm $|||\cdot|||_\Phi$. We will denote by symbol $|||\cdot|||$ any such norm and, according to the basic singular value properties, all such norms satisfy the invariance property $|||UAV||| = |||A|||$ for all unitary *U* and *V*.

Specially well known among those norms are the Schatten *p*-norms defined as $\|A\|_p = (\sum_{i=1}^{\infty} s_i(A)^p)^{1/p}$ for $1 \leq p < \infty$ and represent the norm on the associated ideal ℓ_p known as the Schatten *p*-classes. The Ky-Fan norms defined as $\|A\|_k = \Phi_k(s_i(A)) = \sum_{i=1}^{k} s_i(A)$, $k = 1, 2, \ldots$, represent another interesting family of unitarily invariant norms. The property saying that for all $X \in \ell_\infty$ and $Y \in \ell_\Phi$ with $\|X\|_k \leq \|Y\|_k$ for all $k \geq 1$, we have $X \in \ell_\Phi$ with $|||X||| \leq |||Y|||$ is known as the Ky-Fan dominance property. We note here that the requirement $X \in \ell_\infty$ is just the traditional one, and no harm will be done if we replace it by $X \in B(H)$. Indeed, a simple calculation shows that if $Y \in \ell_\infty$, and $\sum_{i=1}^{k} s_i(X) \leq \sum_{i=1}^{k} s_i(Y)$ for all $k \geq 1$, then $\lim_{n\to\infty} s_n(Y) = 0$ implies that $\lim_{n\to\infty} s_n(X) = 0, i.e., X \in \ell_\infty$.

For a complete account of the theory of norm ideals, the reader is referred to [10], [9] and [1].

NORM INEQUALITIES FOR SELF-ADJOINT

We would like to point out that the Ky-Fan dominance property holds for all bounded operators.

Lemma 2.1: For A in $B(H)$ and $n = 1, 2, \ldots, \dim H$ we have

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$$\sum_{i=1}^{n} si(A) = \sup_{U,e_1,\dots,e_n} \sum_{i=1}^{n} |\langle UAe_i, e_i\rangle| = \sup_{U,e_1,\dots,e_n} \left|\sum_{i=1}^{n} \langle UAe_i, e_i\rangle\right|, \quad (2)$$

where supremum is taken over all unitary operators U on H and all orthonormal systems $e_1,\dots,e_n$ in H.

Proof: The proof differs from [10] (which asserts the same for compact operators), in showing that whenever $s_\infty(A) > 0$ and $m = \dim H_0 < \infty (H_0 = E_{|A|}[s_\infty(A), \|A\|]H$, $E_{|A|}$ is a spectral measure associated to $|A|$), then some right hand side sums can majorize $\sum_{i=1}^{n} s_i(A) - \varepsilon$ for all $\varepsilon > 0$ and $n > m$. But that will really do if for $\delta = \min\{\varepsilon / n, s_\infty(A)\}$ we choose $\{e_1,\dots,e_m\}$ and $\{e_{m+1},\dots,e_n\}$ to be respectively the eigenvectors of $|A|$ in H_0 and any orthonormal system in $E_{|A|}(s_\infty(A)-\delta, s_\infty(A))H$, while U is any unitary operator satisfying $UVe_i = e_i$ for $1 \le i \le n$ (V is from the polar decomposition $A = V|A|$).

In the sequel, a function f satisfying $f(a+b-t) = f(t)$ for all $t \in [a,b]$ will be called symmetric on $[a,b]$. The following lemma generalizes to Lemma (1.1) and a famous Heinz inequality from [5] (see also [12]).

Lemma 2.2: For self-adjoint A and B in $B(H)$ and an arbitrary $X \in B(H)$, for all real $p \ge 1$ and all unitarily invariant norms $|||\cdot|||$, the function

$$f(s) = ||| |A|^{s-1} AX |B|^{p-s} + |A|^{p-s} XB |B|^{s-1} |||$$

is convex and symmetric on $[0, p]$, non-increasing on $[0, p/2]$ and non-decreasing on $[p/2, p]$.

Proof: To show that f is symmetric, we will use the polar decomposition to represent A and B as $A = U|A|$ and $B = V|B|$, with U and V commuting with A and B respectively and satisfying $U^2 = V^2 = 1$. Hence

$$f(p-s) = ||| |A|^{p-s} UX |B|^{s} + |A|^{s} XV |B|^{p-s} |||$$

$$= ||| U(|A|^{s} UX |B|^{p-s} + |A|^{p-s} XV |B|^{s})V ||| = f(s) \quad (3)$$

Next, we show that

$$f\left(\frac{s+t}{2}\right) \le \frac{f(s)+f(t)}{2} \quad (4)$$

for all $1 \le s < t \le p$. Indeed, because of

$$f\left(\frac{s+t}{2}\right) = ||| |A|^{(t-s)/2} (|A|^{s-1} AX |B|^{p-t} + |A|^{p-t} XB |B|^{s-1}) |B|^{(t-s)/2} |||,$$

according to Theorem (1.2) and (3) we have that

$$2f\left(\frac{s+t}{2}\right) \le ||| |A|^{t-s} (|A|^{s-1} AX |B|^{p-t} + |A|^{p-t} XB |B|^{s-1}) |B|$$

$$+(|A|^{s-1} AX |B|^{p-t} + |A|^{p-t} XB|B|^{s-1})|B|^{t-s} |||$$

$$\leq ||| |A|^{p-s} XB|B|^{s-1} + |A|^{s-1} AX |B|^{p-s} |||$$

$$+ ||| |A|^{t-1} AX |B|^{p-t} + |A|^{p-t} XB|B|^{t-1} |||= f(s) + f(t)$$

An immediate consequence of (4) is

$$f(\alpha s + (1-\alpha)t) \leq \alpha f(s) + (1-\alpha) f(t) \qquad (5)$$

for all rational $0 \leq \alpha \leq 1$. For an arbitrary $\alpha \in [0,1]$, we choose a sequence of rational $\alpha_n \in [0,1]$ such that $\lim_{n\to\infty} \alpha_n = \alpha$. Having that the operator valued function

$$g(s) = |A|^{s-1} AX |B|^{p-s} + |A|^{p-s} XB|B|^{s-1}$$

is strongly, and therefore weakly continuous, we get

$$f(\alpha s + (1-\alpha)t) = ||| w - \lim_{n\to\infty} g(\alpha_n s + (1-\alpha_n)t) |||$$

$$\leq \liminf_{n\to\infty} ||| g(\alpha_n s + (1-\alpha_n)t) |||= \liminf_{n\to\infty} f(\alpha_n s + (1-\alpha_n)t)$$

$$\leq \liminf_{n\to\infty} (\alpha_n f(s) + (1-\alpha_n) f(t)) = \alpha f(s) + (1-\alpha) f(t), \qquad (6)$$

Because $||| Y ||| \leq \liminf_{n\to\infty} ||| Y_n |||$ whenever $Y_n \to Y$ weakly in $B(H)$. This follows from the well-known fact that

$$||| Y |||_\Phi = \sup\{|tr(YZ)| : Z \text{ is of finite rank and } ||| Z |||_{\Phi'} \leq 1\}$$

for conjugate gauge functions Φ and Φ'. So (5) holds for all $s, \in [0, p]$ and for all $\alpha \in [0,1]$. According to (3) and (6) f is convex and symmetric on $[0, p]$, and therefore non-increasing on $[0, p/2]$ because

$$f(t) = f\left(\frac{p-s-t}{p-2s} s + \frac{t-s}{p-2s}(p-s)\right)$$

$$\leq \frac{p-s-t}{p-2s} f(s) + \frac{t-s}{p-2s} f(p-s) = f(s),$$

for all $0 \leq s < t \leq p/2$. Similarly, $f(t) \geq f(s)$ for all $p/2 \leq s < t \leq p$, and this ends the proof.

Theorem 2.1: If X and some self-adjoint A and B are in $B(H)$, then

$$||| |AX + XB|^p ||| \leq 2^{p-1} \|X\|^{p-1} ||| |A|^{p-1} AX + XB|B|^{p-1} |||$$

for all real $p \geq 3$ and for all unitarily invariant norms $||| \cdot |||$.

Proof: First, let us consider the particular case: $A = B$, $X = X^*$ and

$||| \cdot |||= \|\cdot\|_K$. We may also suppose $\|X\| \leq 1$, with no loss of generality. An

application of Theorem (2.1) (for $f(p) \geq f(1)$) gives

$$2^{p-1} \| |A|^{p-1} AX + XA|A|^{p-1} \|_k$$

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$$\geq 2^{p-2} \| |A|^{p-1} AX + XA|A|^{p-1} \|_k + 2^{p-2} \| AX|A|^{p-1} + |A|^{p-1} XA \|_k ,$$

and hence

$$2^{p-1} \| |A|^{p-1} AX + XA|A|^{p-1} \|_k$$
$$\geq 2^{p-2} \| |A|^{p-1} (AX + XA) + (AX + XA)|A|^{p-1} \|_k \quad (7)$$

For self-adjoint $A' \in B(H) let E_{A'}$ be its associated spectral measure and let $H_e = E_{A'}(-s_\infty(A'), s_\infty(A'))$. For $m = \dim H \ominus H_e let \lambda_1(A'), \ldots, \lambda_m(A')$ be the eigenvalues of A' in $H \boxminus H_e$, arranged by its non-increasing modulus, with their multiplicities counted. If $m < \infty$, for all $n > m$ let $\lambda_n(A') = s_\infty(A')$ if $s_\infty(A') \in \sigma_{ess}(A')$ and $\lambda_n(A') = -s_\infty(A')$ otherwise. Combining the eigen-vectors of A' in $H \boxminus H_e$ and elements of H_e, we can choose a sequence of orthonormal systems $\left\{e_1^{(n)}, \ldots, e_k^{(n)}\right\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \left\| A' e_i^{(n)} - \lambda_i(A') e_i^{(n)} \right\| \to 0 \text{ for all } 1 \leq i \leq k \quad (8)$$

Specially, for $A' + AX + XA$ it follows from Lemma(2.1), (7) and (8) that

$$2^{p-1} \| |A|^{p-1} AX + XA|A|^{p-1} \|_k$$
$$\geq 2^{p-2} \limsup_{n\to\infty} \sum_{i=1}^{k} \left| \left\langle (|A|^{p-1} (AX + XA) + (AX + XA)|A|^{p-1}) e_i^{(n)}, e_i^{(n)} \right\rangle \right|$$
$$= 2^{p-1} \limsup_{n\to\infty} \sum_{i=1}^{k} \left| \operatorname{Re} \left\langle (AX + XA) e_i^{(n)}, |A|^{p-1} e_i^{(n)} \right\rangle \right|$$
$$= 2^{p-1} \limsup_{n\to\infty} \sum_{i=1}^{k} |\lambda_i(A')| \left\langle |A|^{p-1} e_i^{(n)}, e_i^{(n)} \right\rangle . \quad (9)$$

Spectral representation and Jensen inequality give

$$\left\langle |A|^{p-1} e_i^{(n)}, e_i^{(n)} \right\rangle = \int_0^{+\infty} t^{p-1} d\mu_{e_i^{(n)}}(t) \geq \left(\int_0^{+\infty} t^{p-1} d\mu_{e_i^{(n)}}(t) \right)^{(p-1)/2}$$
$$= \left\langle |A|^2 e_i^{(n)}, e_i^{(n)} \right\rangle^{(p-1)/2} = \| A e_i^{(n)} \|^{p-1}$$
$$\geq \| XA e_i^{(n)} \|^{p-1} \geq \left| \operatorname{Re} \left\langle XA e_i^{(n)}, e_i^{(n)} \right\rangle \right|^{p-1}$$
$$= 2^{p-1} \left| \left\langle (AX + XA) e_i^{(n)}, e_i \right\rangle \right|^{p-1} \to 2^{1-p} |\lambda_i(A')|^{p-1}$$

for $1 \leq i \leq k$, and therefore

$$2^{p-1} \| |A|^{p-1} AX + XA|A|^{p-1} \|_k \geq \sum_{i=1}^{k} |\lambda_i(A')|^p = \| |AX + XA|^p \|_k . \quad (10)$$

Having the proof for our special case completed, for arbitrary self-adjoint A and B in $B(H)$ we will consider the following 2×2 self-adjoint operator matrices

$$C=\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \quad \text{and} \quad Y=\begin{bmatrix} 0 & X \\ X^{*} & 0 \end{bmatrix}$$

acting on $H \oplus H$. A straightforward calculation gives

$$|C|^{p-1}CY+YC|C|^{p-1} = \begin{bmatrix} 0 & |A|^{p-1}AX+XB|B|^{p-1} \\ (|A|^{p-1}AX+XB|B|^{p-1})^{*} & 0 \end{bmatrix}$$

and

$$|CY+YC|^{p}=\begin{bmatrix} |(AX+XB)^{*}|^{p} & 0 \\ 0 & |(AX+XB)|^{p} \end{bmatrix}.$$

As noted in [129]

$$s_{2i-1}(CY+YC)=s_{2i}(CY+YC)=s_{i}(AX+XB),$$

and also

$$s_{2i-1}(|C|^{p-1}CY+YC|C|^{p-1})=s_{2i}(|C|^{p-1}CY+YC|C|^{p-1}) = s_{i}(|A|^{p-1}AX+XB|B|^{p-1}).$$

Applying (10) to self-adjoint C and Y we get

$$2^{p-1}\,\|\,|A|^{p-1}AX+XA|A|^{p-1}\,\|_{k}=2^{p-2}\sum_{j=1}^{2k}s_{i}(|C|^{p-1}CY+YC|C|^{p-1})$$
$$\geq 2^{-1}\sum_{j=1}^{2k}s_{j}^{p}(CY+YC)$$
$$=\sum_{i=1}^{k}s_{i}^{p}(AX+XB)=\|\,|AX+XB|^{P}\,\|_{k}.$$

Now, by the Ky-Fan dominance property we conclude that this inequality also holds for all unitarily invariant norms $|||\cdot|||$.

The preceding theorem can be reformulated as follows.

Theorem 2.2: If X and self-adjoint A and B are in $B(H)$, then for all real $0\leq\alpha\leq 1/3$ then

$$|||\,|A|A|^{\alpha-1}X+XB|B|^{\alpha-1}|^{1/\alpha}\,|||\leq 2^{1/\alpha-1}\,\|X\|^{1/\alpha-1}\,|||AX+XB|||$$

for all unitarily invariant norms $|||\cdot|||$.

Proof: If we denote $C=A|A|^{\alpha-1}$ and $D=D|B|^{\alpha-1}$, then C and D are bounded operators satisfying $|C|=|A|^{\alpha}$ and $|D|=|B|^{\alpha}$. Therefore $|A|=|C|^{1/\alpha}$ and $A=C|C|^{1/\alpha-1}$, and similarly $B=D|D|^{1/\alpha-1}$. An application of Theorem (2.1) to C, D and $1/\alpha$ gives the desired conclusion.

Next, we will show that for $X = 1$ the requirement $P \geq 3$ can be relaxed to $P \geq 2$, even for arbitrary bounded operators A and B. So, we present the following perturbation inequality for bounded operators.

Theorem 2.3: For A and B in $B(H)$ and real $P \geq 2$ we have

$$|||\,|A-B|^P\,||| \leq 2^{P-1} |||\,A|A|^{P-1} - B|B|^{P-1}\,|||$$

for all unitarily invariant norms $|||\cdot|||$.

Proof: We will first treat a case with self-adjoint A and B and the Ky-Fan norms $\|\cdot\|_k$. If we define $|A' = A - B, |\lambda_1(A')| \geq |\lambda_2(\lambda')| \geq \cdots$ and $\{e_1^{(n)}, \dots, e_k^{(n)}\}_{n=1}^{\infty}$ to be as in the proof of Theorem (2.1), then we similarly get

$$\begin{aligned} &2^{p-1} \| A|A|^{p-1} - B|B|^{p-1} \|_k \\ &\geq 2^{p-2} \| |A|^{p-1}(A-B) + (A-B)|B|^{p-1} \|_k \\ &\geq 2^{p-2} \limsup_{n\to\infty} \sum_{i=1}^{k} |\lambda_i(A-B)| (\langle |A|^{p-1} e_i^{(n)}, e_i^{(n)} \rangle \qquad (11) \\ &+ \langle |B|^{p-1} e_i^{(n)}, e_i^{(n)} \rangle). \end{aligned}$$

Jensen inequality shows that

$$\begin{aligned} &\langle |A|^{p-1} e_i^{(n)}, e_i^{(n)} \rangle + \langle |B|^{p-1} e_i^{(n)}, e_i^{(n)} \rangle \\ &\geq \langle |A| e_i^{(n)}, e_i^{(n)} \rangle^{P-1} + \langle |B| e_i^{(n)}, e_i^{(n)} \rangle^{P-1} \\ &\geq |\langle A e_i^{(n)}, e_i^{(n)} \rangle|^{P-1} + |\langle B e_i^{(n)}, e_i^{(n)} \rangle|^{P-1} \\ &\geq 2^{2-P} |\langle (A-B) e_i^{(n)}, e_i^{(n)} \rangle|^{P-1} \to 2^{2-P} |\lambda_i(A-B)|^{P-1} \end{aligned}$$

for $1 \leq i \leq k$ as $n \to \infty$, which together with (11) implies

$$2^{p-1} \| A|A|^{p-1} - B|B|^{p-1} \|_k \geq \| |A-B|^p \|_k . \qquad (12)$$

For arbitrary A and B in $B(H)$ we consider self-adjoint operator matrices

$$C = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}.$$

A straightforward calculation gives

$$\begin{aligned} &C|C|^{p-1} - D|D|^{p-1} \\ &= \begin{bmatrix} 0 & A|A|^{p-1} - B|B|^{p-1} \\ A^*|A^*|^{p-1} - B^*|B^*|^{p-1} & 0 \end{bmatrix} \qquad (13) \end{aligned}$$

and we have to note that $A^* | A^* |^{p-1} - B^* | B^* |^{p-1} = (A | A |^{p-1} - B | B |^{p-1})^*$ simply by the fact that both sides of (13) are self-adjoint. According to [18]

We conclude

$$s_{2i-1}(C-D) = S_{2i}(C-D) = s_i(A-B),$$

and also

$$\begin{aligned} s_{2i-1}(C | C |^{p-1} - D | D |^{p-1}) &= S_{2i}(C | C |^{p-1} - D | D |^{p-1}) \\ &= s_i(A | A |^{p-1} - B | B |^{p-1}). \end{aligned}$$

As (12) holds for all natural k, we get

$$\begin{aligned} 2^{p-1} \| A | A |^{p-1} - B | B |^{p-1} \|_k &= \sum_{i=1}^{k} s_i(A | A |^{p-1} - B | B |^{p-1}). \\ &= 2^{p-1} \sum_{i=1}^{k} s_i(C | C |^{p-1} - D | D |^{p-1}) \\ &\geq 2^{p-1} \sum_{i=1}^{k} s_i^p(C-D) = \| \left| A-B \right|^P \|_k . \end{aligned}$$

To conclude the proof, we just have to invoke the Ky-Fan dominance property for bounded operators.

Similarly as it was done in Theorem (2.2), we can give the following reformulation for the previous theorem.

Theorem 2.4: For A and B be in $B(H)$ and all real $0 \leq \alpha \leq 1/2$ we have

$$||| \left| A | A |^{\alpha-1} - B | B |^{\alpha-1} \right|^{1/\alpha} ||| \leq 2^{1/\alpha-1} ||| A - B |||$$

for all unitarily invariant norms $||| \cdot |||$.

Constants 2^{p-1} and $2^{1/\alpha-1}$ appearing in previous theorems are sharp, as the simple examples *A*=*B*=*X* and $A = -B = 1$ show. Comparing this with constant 1 obtained in [19], we see that Theorems (2.1)-(2.4) extend the corresponding real and complex numbers inequalities to norm inequalities for self-adjoint and bounded operators, just like in [19] did for the difference of positive operators.

In order to complete the above theorems, we give the following.

Theorem 2.5: For *A* and *B* be in *B*(*H*), all real $p \geq 1$ and $0 \leq \alpha \leq 1$ there holds

$$||| \left| \alpha A + (1-\alpha) B \right|^p ||| \leq ||| \alpha | A |^p + (1-\alpha) | B |^p |||$$

for all unitarily invariant norms $||| \cdot |||$.

Proof: Instead of repeating a quite analogous proof, we will just present its essentially different part, saying that for self-adjoint *A* and *B* we have

$$\| \alpha | A |^p + (1-\alpha) | B |^p \|_k \geq \sum_{i=1}^{k} (\alpha \langle | A |^p e_i, e_i \rangle + (1-\alpha) \langle | B |^p e_i, e_i \rangle$$

$$\geq \sum_{i=1}^{k} (\alpha\left|\langle Ae_i, e_i\rangle\right|^p + (1-\alpha)\left|\langle Be_i, e_i\rangle\right|^p)$$

$$\geq \sum_{i=1}^{k} \left|\langle (\alpha A + (1-\alpha)B)e_i, e_i\rangle\right|^p, \qquad (14)$$

according to the convexity of the function $t \to |t|^p$ for $p \geq 1$ and real *t*. The right-hand side of (14) is approximately $\| \left|\alpha A + (1-\alpha)B\right|^p \|_k$ for suitably chosen $\{e_i\}$, that allows us to end this proof, in which, specifically, and the use of Lemma (2.1) has not been required.

MEANS INEQUALITIES FOR ELEMENTARY OPERATORS

We start with the basic Cauchy-Schwarz norm inequality for normal elementary operators. The following theorem extends Theorem 2.5.1 of [8].

Theorem 3.1: If $\sum_{n=1}^{\infty} C_n^* C_n \leq 1, \sum_{n=1}^{\infty} C_n C_n^* \leq 1, \sum_{n=1}^{\infty} D_n^* D_n \leq 1$ and $\sum_{n=1}^{\infty} D_n D_n^* \leq 1$ for some operator families $\{C_n\}_{n=1}^{\infty}$ and $\{D_n\}_{n=1}^{\infty}$, then also $\sum_{n=1}^{\infty} C_n Y D_n \in \mathcal{C}_{\|\cdot\|}$ whenever $Y \in \mathcal{C}_{\|\cdot\|}$ for some unitarily invariant norm $|||\cdot|||$, and moreover

$$\left|\left|\left|\sum_{n=1}^{\infty} C_n Y D_n\right|\right|\right| \leq |||Y||| \qquad (15)$$

Proof: For arbitrary f and g in H a straightforward calculation gives

$$\left|\left\langle\left(\sum_{n=1}^{\infty} C_n Y D_n\right) f, g\right\rangle\right| \leq \sum_{n=1}^{\infty} \|Y\| \|D_n f\| \|C_n^* g\|$$

$$\leq \|Y\| \left(\sum_{n=1}^{\infty} \|D_n f\|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \|C_n^* g\|^2\right)^{\frac{1}{2}}$$

$$= \|Y\| \left\langle \sum_{n=1}^{\infty} D_n^* D_n f, f\right\rangle^{\frac{1}{2}} \left\langle \sum_{n=1}^{\infty} C_n C_n^* g, g\right\rangle^{\frac{1}{2}}$$

$$= \|Y\| \left\|\left(\sum_{n=1}^{\infty} C_n C_n^*\right)^{\frac{1}{2}} g\right\| \left\|\left(\sum_{n=1}^{\infty} D_n^* D_n\right)^{\frac{1}{2}} f\right\| \leq \|Y\| \|f\| \|g\|$$

from which we conclude that

$$\left\|\sum_{n=1}^{\infty} C_n Y D_n\right\| \leq \|Y\| \qquad (16)$$

Therefore, for all $N = 1, 2, ...,$ for $Y \in \mathcal{C}_1$ and for all $W \in B(H)$ we have

$$\left|tr\left(\sum_{n=1}^{N} C_n Y D_n W^*\right)\right| = \left|tr\left(Y\left(\sum_{n=1}^{N} C_n^* W D_n^*\right)^*\right)\right|$$

$$\leq \|Y\|_1 \left\| \sum_{n=1}^{\infty} C_n^* W D_n^* \right\| \leq \|Y\|_1 \|W\|,$$

according to (16), from which we deduce that

$$\left\| \sum_{n=1}^{N} C_n Y D_n \right\|_1 \leq \|Y\|_1 \quad (17)$$

If $Y \in \mathcal{C}_\infty$ let $Y = \sum_{n=1}^{\infty} s_n(Y)\langle \cdot, e_n \rangle f_n$ be a singular value decomposition for some orthonormal systems $\{e_n\}$ and $\{f_n\}$.For all $k \geq 2$ we introduce operators

$$Z = \sum_{n=1}^{k-1} (s_n(Y) - s_{n+1}(Y)) \sum_{j=1}^{n} \langle \cdot, e_j \rangle f_j$$

$$V = s_k(Y) \sum_{n=1}^{k} \langle \cdot, e_n \rangle f_n + \sum_{n=k+1}^{\infty} s_n(Y) \langle \cdot, e_n \rangle f_n$$

We see that

$$Z = \sum_{n=1}^{k-1} \sum_{j=1}^{n} (s_n(Y) - s_{n-1}(Y)) \langle \cdot, e_j \rangle f_j$$

$$= \sum_{j=1}^{k} (s_j(Y) - s_k(Y)) \langle \cdot, e_j \rangle f_j$$

$$= \sum_{n=1}^{k} s_n(Y) \langle \cdot, e_n \rangle f_n + s_k(Y) \sum_{n=1}^{k} \langle \cdot, e_n \rangle f_n = Y - V$$

We can also note that $s_1(V) = \cdots = s_k(V) = s_k(Y)$ due to orthogonality of the systems $\{e_n\}$ and $\{f_n\}$.That will allows us to conclude that for all Ky Fan k-norms we have

$$\left\| \sum_{n=1}^{N} C_n Y D_n \right\|_k \leq \left\| \sum_{n=1}^{N} C_n Z D_n \right\| + \left\| \sum_{n=1}^{N} C_n V D_n \right\|$$

$$\leq \|Z\|_1 + k \left\| \sum_{n=1}^{N} C_n Z D_n \right\|_\infty \quad (18)$$

$$\leq (s_n(Y) - s_{n-1}(Y)) \sum_{j=1}^{n} \left\| \langle \cdot, e_j \rangle f_j \right\|_\infty + k \|V\|_\infty \quad (19)$$

$$\leq \sum_{n=1}^{k-1} n(s_n(Y) - s_{n-1}(Y)) + k s_k(Y) = \sum_{n=1}^{k} s_n(Y) = \|Y\|_k \quad (20)$$

with (18) following from (17) and (19) from (16).

Moreover, if Y is in $\mathcal{C}_\infty$ then also $\sum_{n=1}^{\infty} C_n Y D_n \in \mathcal{C}_\infty$ Indeed, elementary operators $R_N(Y) = \sum_{n=1}^{N} C_n Y D_n$ acting on $\mathcal{C}_\infty^{(K)}$ represent a bounded family, because $\|R_N(Y)\|_k \leq \|Y\|_k$ for all $Y \in \mathcal{C}_\infty$ by (20). Also, for one dimensional operators $f \otimes g$ and $M > N$ we have

$$\|R_M(f \otimes g) - R_N(f \otimes g)\|_k \leq \left\| \sum_{n=N+1}^{M} D_n^* f \otimes C_n g \right\|_1$$

$$\leq \sum_{n=N+1}^{M} \|D_n^* f\| \|C_n g\| \leq \left\| \left(\sum_{n=N+1}^{M} C_n C_n^* \right)^{\frac{1}{2}} g \right\| \left\| \left(\sum_{n=N+1}^{M} D_n^* D_n \right)^{\frac{1}{2}} f \right\|$$

which $\to 0$ as $M, N \to \infty$. Therefore $R_N(Y)$ converge in $C_\infty^{(K)}$ for all finite dimensional Y to a compact operator. By the uniform boundedness principle the same is true for all $Y \in C_\infty^{(K)}$, due to its separability. So (15) holds for all Ky Fan k-norms, and we therefore invoke the Ky Fan dominance property to conclude that (15) holds for all unitarily invariant norms, as required.

In the sequel we will refer to a family $\{A_n\}_{n=1}^{\infty}$ in $B(H)$ as square summable if $\sum_{n=1}^{\infty} \|A_n f\|^2 < \infty$ for all $f \in H$. Though this means just the weak convergence of $\sum_{n=1}^{\infty} A_n^* A_n$, an appeal to the resonance principle shows that $\sum_{n=1}^{\infty} A_n^* A_n$ actually defines a bounded Hilbert space operator, and due to the monotonicity of its partial sums, the convergence is moreover strong. For such families the following Cauchy-Schwarz inequality holds:

Theorem 3.2: For a square summable families $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ of commuting normal operators

$$\left|\left|\left| \sum_{n=1}^{\infty} A_n X B_n \right|\right|\right| \leq \left|\left|\left| \left(\sum_{n=1}^{\infty} A_n^* A_n \right)^{1/2} X \left(\sum_{n=1}^{\infty} B_n^* B_n \right)^{1/2} \right|\right|\right|, \qquad (21)$$

for all $X \in B(H)$ and for all unitarily invariant norms $|||\cdot|||$. If $C_{|||\cdot|||}$ is separable and $X \in C_{|||\cdot|||}$ then the left-hand side sum converges in the norm of this ideal.

Proof: First, we need a suitable factorization for Hilbert space operators A_n and B_n. Let $A = (\sum_{n=1}^{\infty} A_n^* A_n)^{\frac{1}{2}}$ and $B = (\sum_{n=1}^{\infty} B_n^* B_n)^{\frac{1}{2}}$, and let P and Q denote respectively the orthogonal projections on $\overline{R(A)}$ and $\overline{R(B)}$. If for a given $f \in H$ we have that $Pf = \lim_{k\to\infty} Ag_k$ for some sequence $\{g_k\}$ in H, then $\lim_{k\to\infty} A_n g_k$ exists for all $n \geq 1$ and does not depend on the chosen sequence. Indeed,

$$\|A_n g_k - A_n g_l\| \leq \|A(g_k - g_l)\| \to \|Pf - Pf\| = 0$$

as $k, l \to \infty$, and also $\|A_n g_k - A_n g_l\| \leq \|A(g_k - g_l)\| \to 0$ as $k \to \infty$ whenever $\lim_{k\to\infty} Ah_k = Pf$ for some other sequence $\{h_k\}$. Thus we can correctly introduce operators $C_n, n = 1, 2, ...,$ by $C_n f = \lim_{k\to\infty} A_n g_k$, where $\{g_k\}$ is any sequence in H such that $\lim_{k\to\infty} Ag_k = Pf$. Let us note that due to our definition every C_n vanishes on $N(A)$, i.e., $C_n = C_n P$, and also $C_n A = AC_n = A_n$.

Moreover, $\sum_{n=1}^{\infty} C_n^* C_n = P$. Indeed, $\sum_{n=1}^{\infty} C_n^* C_n A^2 = \sum_{n=1}^{\infty} A_n^* A_n = A^2$ implies $\sum_{n=1}^{\infty} C_n^* C_n P = P$, which together with the fact that $C_n(1-P) = 0$ gives the desired conclusion. For all $m, n = 1, 2, ..., C_m^*$ and C_n commute on $R(A^2)$ and $N(A^2)$, and so also on all of H. Thus $\{C_n\}_{n=1}^{\infty}$ is a commuting family of normal contractions which realize the factorizations $C_n A = AC_n = A_n$, with $\sum_{n=1}^{\infty} C_n^* C_n = P$, and which commute with the family $\{A_n\}_{n=1}^{\infty}$. Similarly we get a commuting family $\{D_n\}_{n=1}^{\infty}$ of normal contractions which also commute with $\{B_n\}_{n=1}^{\infty}$ and satisfy $D_nB = BD_n = B_n$ and $\sum_{n=1}^{\infty} D_n^* D_n = Q$.One could easily derive the next explicit formula: $C_n = \overline{A_n A^{\dagger}} = \overline{A^{\dagger} A_n}$, where $A^{\dagger}$ denotes a (densely defined) Moore-Penrose (generalized) inverse for A.

For $Y = AXB \in \mathcal{C}_{|||\cdot|||}$ (there is nothing to prove in the opposite case), an application of Theorem (3.1) gives

$$\left|\left|\left|\sum_{n=1}^{\infty} A_n X B_n\right|\right|\right| = \left|\left|\left|\sum_{n=1}^{\infty} C_n Y D_n\right|\right|\right|$$

$$\leq |||Y||| = \left|\left|\left|\left(\sum_{n=1}^{\infty} A_n^* A_n\right)^{1/2} X \left(\sum_{n=1}^{\infty} B_n^* B_n\right)^{1/2}\right|\right|\right| \tag{22}$$

which proves the first part of theorem.

Finally, if $\mathcal{C}_{|||\cdot|||}$ is separable, then for all $N = 1, 2, ...,$ an application of the just proven part of theorem combined with the arithmetic-geometric means inequality in [17] gives

$$\left|\left|\left|\sum_{n=N}^{\infty} A_n X B_n\right|\right|\right| = \left|\left|\left|\left(\sum_{n=N}^{\infty} A_n^* A_n\right)^{1/2} X \left(\sum_{n=N}^{\infty} B_n^* B_n\right)^{1/2}\right|\right|\right|$$

$$= \left|\left|\left|\left(\sum_{n=N}^{\infty} C_n^* C_n\right)^{1/2} AXB \left(\sum_{n=N}^{\infty} D_n^* D_n\right)^{1/2}\right|\right|\right|$$

$$= \frac{1}{2}\left|\left|\left|\left(\sum_{n=N}^{\infty} C_n^* C_n\right) AXB + AXB \left(\sum_{n=N}^{\infty} D_n^* D_n\right)\right|\right|\right| \tag{23}$$

We see by (22) that $\{\sum_{n=N}^{\infty} C_n^* C_n\}_{N=1}^{\infty}$ and $\{\sum_{n=N}^{\infty} D_n^* D_n\}_{N=1}^{\infty}$ represent bounded sequences of self-adjoint operators which strongly converge to 0 as $N \to \infty$. As $AXB \in \mathcal{C}_{|||\cdot|||}$ which is separable, then the right-hand side of (23) tends to 0 as $N \to \infty$ by [8]. The conclusion follows.

Corollary 3.1: For normal A and B in $B(H)$ and for all real $r \geq 2$.

$$\left|\left|\left|\frac{AX + XB}{2}\right|\right|\right| \leq \left|\left|\left|\left(\frac{1+|A|^r}{2}\right)^{\frac{1}{r}} X \left(\frac{1+|B|^r}{2}\right)^{\frac{1}{r}}\right|\right|\right| \tag{24}$$

as well as

$$\left|\left|\left|\frac{X + AXB}{2}\right|\right|\right| \leq \left|\left|\left|\left(\frac{1+|A|^r}{2}\right)^{\frac{1}{r}} X \left(\frac{1+|B|^r}{2}\right)^{\frac{1}{r}}\right|\right|\right| \tag{25}$$

for all $X \in B(H)$ and for all unitarily invariant norms $|||\cdot|||$.

Proof: $\{A, I\}$ and $\{I, B\}$ are families of normal commuting operators, and so for $r = 2$ the desired conclusion follows by Theorem (3.2) For $r > 2$ the mapping $t \to t^{\frac{2}{r}}$ is operator monotone by a well-known Heinz theorem, and therefore this is an operator concave mapping (see [11]). Specifically, $\frac{1+|A|^2}{2} \le \left(\frac{1+|A|^r}{2}\right)^{\frac{2}{r}}$, from which we obtain

$$\left\|\left(\frac{1+|A|^2}{2}\right)^{\frac{1}{2}}\left(\frac{1+|A|^r}{2}\right)^{-\frac{1}{r}}\right\| \le 1$$

and similarly

$$\left\|\left(\frac{1+|B|^2}{2}\right)^{\frac{1}{2}}\left(\frac{1+|B|^r}{2}\right)^{-\frac{1}{r}}\right\| \le 1$$

Therefore

$$\left\|\left\|\left(\frac{1+|A|^2}{2}\right)^{\frac{1}{2}} X \left(\frac{1+|B|^2}{2}\right)^{\frac{1}{2}}\right\|\right\| \le \left\|\left\|\left(\frac{1+|A|^r}{2}\right)^{\frac{1}{r}} X \left(\frac{1+|B|^r}{2}\right)^{\frac{1}{r}}\right\|\right\|,$$

which completes the proof.

Corollary 3.2: For normal A and B in $B(H)$ the inequality

$$\left\|\left\|\frac{AX + XB}{2}\right\|\right\| \le \||X\||^{1-\frac{1}{r}} \left\|\left\|\frac{|A|^r X + X|B|^r}{2}\right\|\right\|^{\frac{1}{r}} \qquad (26)$$

holds for all real $r \ge 2$,for all unitarily invariant norms $\||\cdot\||$ and for all $X \in C_{\||\cdot\||}$

Proof: By Corollary (3.1), for all $t > 0$,

$$\left\|\left\|\frac{AX + XB}{2}\right\|\right\| = t^{-1}\left\|\left\|\frac{tAX + XtB}{2}\right\|\right\|$$

$$\le t^{-1}\left\|\left\|\left(\frac{1+|tA|^r}{2}\right)^{\frac{1}{r}} X \left(\frac{1+|tB|^r}{2}\right)^{\frac{1}{r}}\right\|\right\|$$

and therefore

$$\left\|\left\|\frac{AX + XB}{2}\right\|\right\| \le t^{-1}\||X\||^{1-\frac{2}{r}} \left\|\left\|\left(\frac{1+|tA|^r}{2}\right)^{\frac{1}{2}} X \left(\frac{1+|tB|^r}{2}\right)^{\frac{1}{2}}\right\|\right\|,$$

by [7], because $\frac{2}{r} < 1$.Therefore, the arithmetic-geometric mean inequality implies

$$\left\|\left|\frac{AX+XB}{2}\right|\right\| \le \frac{1}{2t}\left\|\left|X\right|\right\|^{1-\frac{2}{r}}\left\|\left|\frac{1+|tA|^r}{2}X+X\frac{1+|tB|^r}{2}\right|\right\|^{\frac{2}{r}}$$

$$\le \frac{1}{2}\left\|\left|X\right|\right\|^{1-\frac{2}{r}}\left(t^{-\frac{r}{2}}\left\|\left|X\right|\right\|+t^{\frac{r}{2}}\left\|\left|\frac{|A|^r X+X|B|^r}{2}\right|\right\|\right)^{\frac{2}{r}} \qquad (27)$$

As the right-hand side equals $\left\|\left|X\right|\right\|^{1-\frac{1}{r}}\left\|\left|\frac{|A|^r X+X|B|^r}{2}\right|\right\|^{\frac{1}{r}}$, which attains its minimum for $t=\left\|\left|X\right|\right\|^{\frac{1}{r}}\left\|\left|\frac{|A|^r X+X|B|^r}{2}\right|\right\|^{-\frac{1}{r}}$, the conclusion follows.

Theorem 3.3: For normal contractions A and B the inequality

$$\left\|\left|(I-A^*A)^{\frac{1}{2}}X(I-B^*B)^{\frac{1}{2}}\right|\right\| \le \left\|\left|X-AXB\right|\right\|, \qquad (28)$$

holds for all $X\in B(H)$ and for all unitarily invariant norms $\left\|\left|\cdot\right|\right\|$.

Proof: First, we note that $s-\lim_{n\to\infty}A^n(I-A^*A)^{\frac{1}{2}}=0$ .Indeed, by a spectral theorem, for every $f\in H$ there is a positive, finite Borel measure μ concentrated on $D=\{z\in C:|z|\le 1\}$ such that

$$\left\|A^n(I-A^*A)^{\frac{1}{2}}f\right\|^2=\int_D |z|^{2n}\left(1-|z|^2\right)d\mu_f(z)$$

whence the desired conclusion follows by Lebesgue's dominating convergence theorem. Therefore

$$w-\lim_{n\to\infty}\left(I-A^*A\right)^{\frac{1}{2}}\left(X-A^nXB^n\right)\left(I-B^*B\right)^{\frac{1}{2}}=\left(I-A^*A\right)^{\frac{1}{2}}X\left(I-B^*B\right)^{\frac{1}{2}}$$

So by Theorem (2.3.2) we get

$$\left\|\left|\left(I-A^*A\right)^{\frac{1}{2}}X\left(I-B^*B\right)^{\frac{1}{2}}\right|\right\|$$

$$=\left\|\left|\lim_{n\to\infty}\left(I-A^*A\right)^{\frac{1}{2}}\left(X-A^nXB^n\right)\left(I-B^*B\right)^{\frac{1}{2}}\right|\right\|$$

$$=\left\|\left|\sum_{k=0}^{\infty}\left(I-A^*A\right)^{\frac{1}{2}}A^k\left(X-AXB\right)B^k\left(I-B^*B\right)^{\frac{1}{2}}\right|\right\|$$

$$\le\left\|\left|\left(\sum_{k=0}^{\infty}(I-|A|^2)|A|^{2k}\right)^{\frac{1}{2}}\left(X-AXB\right)\left(\sum_{k=0}^{\infty}|B|^{2k}(I-|B|^2)\right)^{\frac{1}{2}}\right|\right\|$$

$$=\left\|\left|(I-P)(X-AXB)(I-Q)\right|\right\|\le\left\|\left|X-AXB\right|\right\|, \qquad (29)$$

where P and Q are the orthogonal projections on $\ker(I-A^*A)$ and $\ker(I-B^*B)$ respectively. This concludes the proof.

MAIN RESULTS

Theorem 4.1:

If A and B are self-adjont operators and an arbitrary X are in $B(H)$ where $AX = XB$ then

$$\left\|\left| \, |AX+XB|^2 \, \right|\right\| \le 2\left\|\left| \, |AX|^2+|XB|^2 \right|\right\|$$

Proof:

$$\left\|\left| \, |AX+XB|^2 \, \right|\right\|$$

$$=\left\|\left|(AX+XB)(AX+XB)^*\right|\right\| = \left\|\left|(AX+XB)(A^*X^*+X^*B^*)\right|\right\|$$

$$=\left\|\left|AXA^*X^*+AXX^*B^*+XBA^*X^*+XBX^*B^*\right|\right\|$$

$$\le\left\|\left|AXA^*X^*+XBX^*B^*\right|\right\|+\left\|\left|AXX^*B^*+XBA^*X^*\right|\right\|$$

$$=\left\|\left| \, |AX|^2+|XB|^2\right|\right\|+\left\|\left|XBX^*B^*+AXA^*X^*\right|\right\|$$

$$=\left\|\left| \, |AX|^2+|XB|^2\right|\right\|+\left\|\left| \, |XB|^2+|AX|^2\right|\right\|$$

$$=2\left\|\left| \, |AX|^2+|XB|^2\right|\right\|$$

Theorem 4.2: For self-adjoint normal contraction operators A and B

$$\left\|\left|(I-|A^*|)X^2(I-|B^*|)\right|\right\| \le \left\|\left|X-A^*XB^*\right|\right\|^2$$

holds for all $X \in B(H)$.

Proof: Similarly as in the proof of theorem (2.3.5) for $S-\lim_{n\to\infty} A^{*^n}\left(I-|A^*|^2\right)^{\frac{1}{2}}=0$, therefore

$$w-\lim_{n\to\infty}\left(I-|A^*|^2\right)^{\frac{1}{2}}\left(X-A^{*^n}XB^{*^n}\right)\left(I-|B^*|^2\right)^{\frac{1}{2}}$$

$$=\left(I-|A^*|^2\right)^{\frac{1}{2}}X\left(I-|B^*|^2\right)^{\frac{1}{2}}$$

We get

$$\left\|\left|\left(I-|A^*|^2\right)X^2\left(I-|B^*|^2\right)\right|\right\|^{1/2}$$

$$= \left\| \lim_{n\to\infty} \left(I - \left| A^* \right|^2 \right)^{\frac{1}{2}} \left(X - A^{*^n} X B^{*^n} \right) \left(I - \left| B^* \right|^2 \right)^{\frac{1}{2}} \right\|$$

$$= \left\| \left(I - \left| A^* \right|^2 \right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} A^{*^k} (A^* X B^*) B^{*^k} \right) \left(I - \left| B^* \right|^2 \right)^{\frac{1}{2}} \right\|$$

$$\leq \left\| \left(\sum_{k=0}^{\infty} \left(I - \left| A^* \right|^2 \right) \left| A^{*^{2k}} \right| \right)^{\frac{1}{2}} \left(X - A^* X B^* \right) \left(\sum_{k=0}^{\infty} \left| B^{*^{2k}} \right| \left(I - \left| B^* \right|^2 \right) \right)^{\frac{1}{2}} \right\|$$

$$= \left\| (I - P^*) \left(X - A^* X B^* \right) (I - Q^*) \right\|$$

$$\leq \left\| X - A^* X B^* \right\|$$

Hence

$$\left\| \left(I - \left| A^* \right|^2 \right) X^2 \left(I - \left| B^* \right|^2 \right) \right\| \leq \left\| X - A^* X B^* \right\|^2$$

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