Cauchy-Schwarz and Means Inequalities for Selfadjoint Operator

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ABSTRACT
We determined that for bounded operators $A$ and $B$ on a Hilbert space there holds
\[ \|A - B\|^p \leq 2^{p-1} \|A^{-1} - B^{-1}\| \] for all $p \geq 2$, and if $A$ and $B$ are additionally self-adjoint operators, then
\[ \|AX + XB\|^p \leq 2^{p-1} \|X\|^{p-1} \|A^{-1}AX + XB\| \] for all $p \geq 3$. We proved that if $A$ and $B$ are self-adjoint operators on $B(H)$ where $AX = XB$ then
\[ \|AX + XB\|^2 \leq 2 \|AX\|^2 + \|XB\|^2. \]
Similarly the inequality \[ \|A^2X + B^2X\|^{1/2} \leq \frac{1}{2} \|AX + XB\| \] is known. We proved that for self-adjoint normal contractions operators $A$ and $B$ then
\[ \left\| \left( I - A^* \right)X \left( I - B^* \right) \right\| \leq \left\| X - A^*XB \right\|^2 \] hold, for all $X \in B(H)$.

Keywords: Cauchy-schwarz - selfadjoint operator - Norm Inequalities

PRELIMINARIES
Recently, the following perturbation norm inequality has been established in [2].

**Theorem 1.1:** If $A$ and $B$ are self-adjoint operators in $B(H)$, then for all natural numbers $n$ and every unitarily invariant norm \( \| \cdot \| \),
\[ \|A - B\|_{2n+1} \leq 2^n \|A^{2n+1} - B^{2n+1}\|. \] (1)
This completely resolves the problem raised by Koplienko and others (see [14] and [15]) for estimating $\|A - B\|$ when $A^n - B^n$ is given in a specified norm ideal.

**Lemma 1.1:** If self-adjoint $A$ and $B$ and an arbitrary $X$ are in $B(H)$, then for all non-negative integers $n$ and every unitarily invariant norm $\| \cdot \|$, there holds the following chain inequality
\[ \|A^n (A - B) B^n\| \leq A^{2^{-1}} (A^3 - B^3) B^{n-1} \leq \cdots \leq \|A^{2n+1} - B^{2n+1}\|. \]
This lemma itself was based on the following arithmetic-geometric mean inequality (see [16] and [6]), which we will also need in the sequel.

**Theorem 1.2:** For arbitrary $A$, $B$ and $X$ in $B(H)$, and every unitarily invariant norm $\| \cdot \|$,\[ 2 \|A^*XB\| \leq \|AA^* X + XBB^*\|. \]
We will show that the inequality (1) also holds for all self adjoint derivations $A X - X B$ and for all real $n \geq 0$.

Let $B(H)$ and $\ell_\infty$ denote respectively the space of all bounded and compact linear operators acting on a separable, infinite-dimensional, complex Hilbert space $H$. Following [10], for an arbitrary $A \in B(H)$, let $s_1(A) \geq s_2(A) \geq \cdots$ denote the singular values of $A$, i.e., the eigenvalues of $|A| = (A^* A)^{1/2}$ exceeding the essential norm $\|A\| = s_\infty(A) = \sup \sigma_{\text{ess}}(|A|)$, arranged in a non-increasing order, with their (necessarily finite) multiplicities counted. If necessary, this sequence can always be made infinite by adding $s_n(A) = s_\infty(A)$ for missing $n$. Note that for all bounded $A$ we have $s_\infty(A) = \lim_{n \to \infty} s_n(A)$, while $A \in \ell_\infty$ if and only if $s_\infty(A) = 0$. For the extension of some standard singular value properties to bounded operators see [10] and [13].

Each symmetric gauge function $\Phi$ on sequences (see [10] for definition) gives rise to a unitarily invariant norm on operator ideal $\ell_\Phi$ contained in $\ell_\infty$ which is complete in the topology induced by the norm $\|\cdot\|_{\Phi}$. We will denote by symbol $\|\cdot\|_{\Phi}$ any such norm and, according to the basic singular value properties, all such norms satisfy the invariance property $\|UAV\| = \|A\|$ for all unitary $U$ and $V$.

Specially well known among those norms are the Schatten $p$-norms defined as $\|A\|_p = \left(\sum_{i=1}^{\infty} s_i(A)^p\right)^{1/p}$ for $1 \leq p < \infty$ and represent the norm on the associated ideal $\ell_p$ known as the Schatten $p$-classes. The Ky-Fan norms defined as $\|A\|_k = \Phi_k(s_i(A)) = \sum_{i=1}^{k} s_i(A)$, $k = 1, 2, \ldots$, represent another interesting family of unitarily invariant norms. The property saying that for all $X \in \ell_\infty$ and $Y \in \ell_\Phi$ with $\|X\|_k \leq \|Y\|_k$ for all $k \geq 1$, we have $X \in \ell_\Phi$ with $\|X\|_k \leq \|Y\|_k$ is known as the Ky-Fan dominance property. We note here that the requirement $X \in \ell_\infty$ is just the traditional one, and no harm will be done if we replace it by $X \in B(H)$. Indeed, a simple calculation shows that if $Y \in \ell_\infty$, and $\sum_{i=1}^{k} s_i(X) \leq \sum_{i=1}^{k} s_i(Y)$ for all $k \geq 1$, then $\lim_{n \to \infty} s_n(Y) = 0$ implies that $\lim_{n \to \infty} s_n(X) = 0$, i.e., $X \in \ell_\infty$.

For a complete account of the theory of norm ideals, the reader is referred to [10], [9] and [1].

**NORM INEQUALITIES FOR SELF-ADJOINT**

We would like to point out that the Ky-Fan dominance property holds for all bounded operators.

**Lemma 2.1:** For $A$ in $B(H)$ and $n = 1, 2, \ldots$, $\dim H$ we have
\[
\sum_{i=1}^{n} s_i(A) = \sup_{U, e_1, \ldots, e_n} \left| \sum_{i=1}^{n} \left( U A e_i, e_i \right) \right| = \sup_{U, e_1, \ldots, e_n} \left| \sum_{i=1}^{n} \left( U A e_i, e_i \right) \right|, \quad (2)
\]

where supremum is taken over all unitary operators \( U \) on \( H \) and all orthonormal systems \( e_1, \ldots, e_n \) in \( H \).

**Proof:** The proof differs from [10] (which asserts the same for compact operators), in showing that whenever \( s_\omega(A) > 0 \) and \( m = \dim H_0 < \infty \) (\( H_0 = E_{|A|}[s_\omega(A)], \|A\|H \), \( E_{|A|} \) is a spectral measure associated to \( |A| \), then some right hand side sums can majorize \( \sum_{i=1}^{n} s_i(A) - \varepsilon \) for all \( \varepsilon > 0 \) and \( n > m \). But that will really do if for \( \delta = \min \{ \varepsilon / n, s_\omega(A) \} \) we choose \( \{ e_1, \ldots, e_n \} \) and \( \{ e_{m+1}, \ldots, e_n \} \) to be respectively the eigenvectors of \( |A| \) in \( H_0 \) and any orthonormal system in \( E_{|A|}(s_\omega(A) - \delta, s_\omega(A))H \), while \( U \) is any unitary operator satisfying \( U Ve_i = e_i \) for \( 1 \leq i \leq n \) (\( V \) is from the polar decomposition \( A = V |A| \)).

In the sequel, a function \( f \) satisfying \( f(a + b - t) = f(t) \) for all \( t \in [a,b] \) will be called symmetric on \([a,b] \). The following lemma generalizes to Lemma (1.1) and a famous Heinz inequality from [5] (see also [12]).

**Lemma 2.2:** For self-adjoint \( A \) and \( B \) in \( B(H) \) and an arbitrary \( X \in B(H) \), for all real \( p \geq 1 \) and all unitarily invariant norms \( \| \cdot \|_p \), the function

\[
f(s) = \| \| A \|^{-1} AX \|_p B \|_p^{-1} + \| A \|^{-1} XB \|_p^{-1} \|
\]

is convex and symmetric on \([0,p] \), non-increasing on \([0,p/2] \) and non-decreasing on \([p/2,p] \).

**Proof:** To show that \( f \) is symmetric, we will use the polar decomposition to represent \( A \) and \( B \) as \( A = U |A| \) and \( B = V |B| \), with \( U \) and \( V \) commuting with \( A \) and \( B \) respectively and satisfying \( U^2 = V^2 = 1 \). Hence

\[
f(p - s) = \| \| A \|^{-1} UX \|_p B \|_p^{-1} + \| A \|^{-1} XV \|_p B \|_p^{-1} \|
\]

\[
= \| U (|A|^{-1} UX |B|^{-1} + |A|^{-1} XV |B|^{-1}) V \|_p \| = f(s) \quad (3)
\]

Next, we show that

\[
f\left( \frac{s + t}{2} \right) \leq \frac{f(s) + f(t)}{2} \quad (4)
\]

for all \( 1 \leq s < t \leq p \). Indeed, because of

\[
f\left( \frac{s + t}{2} \right) = \| |A|^{-1/2} (|A|^{-1} AX |B|^{-1} + |A|^{-1} XB |B|^{-1}) |B|^{-1/2} \|,
\]

according to Theorem (1.2) and (3) we have that

\[
2f\left( \frac{s + t}{2} \right) \leq \| |A|^{-1} AX |B|^{-1} + |A|^{-1} XB |B|^{-1} \| \]
An immediate consequence of (4) is
\[
\left| f(\alpha s + (1 - \alpha)t) - f(s) - (1 - \alpha)f(t) \right| \leq \alpha f(s) + (1 - \alpha)f(t) \tag{5}
\]
for all rational \(0 \leq \alpha \leq 1\). For an arbitrary \(\alpha \in [0,1]\), we choose a sequence of rational \(\alpha_n \in [0,1]\) such that \(\lim_{n \to \infty} \alpha_n = \alpha\). Having that the operator valued function
\[
g(s) = \left| A^{-1} AX |B|^{\alpha-s} + |A|^{\alpha-s} XB|B|^{-\alpha} \right|
\]
is strongly, and therefore weakly continuous, we get
\[
f(\alpha s + (1 - \alpha)t) = \lim_{n \to \infty} g(\alpha_n s + (1 - \alpha_n)t)
\]
\[
\leq \liminf_{n \to \infty} g(\alpha_n s + (1 - \alpha_n)t) = \liminf_{n \to \infty} f(\alpha_n s + (1 - \alpha_n)t)
\]
\[
\leq \liminf_{n \to \infty} (\alpha f(s) + (1 - \alpha)f(t)) = \alpha f(s) + (1 - \alpha)f(t), \tag{6}
\]
Because \(\|Y\| \leq \liminf_{n \to \infty} \|Y_n\| \leq \|Y\| \) whenever \(Y_n \to Y\) weakly in \(B(H)\). This follows from the well-known fact that
\[
\|Y\|_{v} = \sup \{\|tr(YZ)\|: Z \text{ is of finite rank and } \|Z\|_{v'} \leq 1\}
\]
for conjugate gauge functions \(\Phi\) and \(\Phi'\). So (5) holds for all \(s,t \in [0, p]\) and for all \(\alpha \in [0,1]\). According to (3) and (6) \(f\) is convex and symmetric on \([0, p]\), and therefore non-increasing on \([0, p/2]\) because
\[
f(t) = \left( \frac{p-s-t}{p-2s} s + \frac{1-s}{p-2s} (p-s) \right) f(s) + \frac{1-s}{p-2s} f(p-s) = f(s),
\]
for all \(0 \leq s < t \leq p/2\). Similarly, \(f(t) \geq f(s)\) for all \(p/2 \leq s < t \leq p\), and this ends the proof.

**Theorem 2.1:** If \(X\) and some self-adjoint \(A\) and \(B\) are in \(B(H)\), then
\[
\| |AX + XB|^{p}| \| \leq 2^{p-1} \|X\|^{p-1} \| |A|^{p-1} AX + XB|B|^{p-1} \|
\]
for all real \(p \geq 3\) and for all unitarily invariant norms \(\|\cdot\|_\kappa\).

**Proof:** First, let us consider the particular case: \(A = B = X\) and \(\|\cdot\| = \|\cdot\|_\kappa\). We may also suppose \(\|X\| \leq 1\), with no loss of generality. An application of Theorem (2.1) (for \(f(p) \geq f(1)\)) gives
\[
2^{p-1} \| |A|^{p-1} AX + XA|A|^{p-1} \|_\kappa
\]
\[ \geq 2^{p-2} \|A\|^{p-1} AX +XA \|A\|^{p-1} \|_{k} + 2^{p-2} \|AX \|^{p-1} + \|A\|^{p-1} AX \|_{k}, \]

and hence
\[ 2^{p-1} \|A\|^{p-1} AX +XA \|A\|^{p-1} \|_{k} \]

\[ \geq 2^{p-2} \|A\|^{p-1} (AX +XA) + (AX +XA) \|A\|^{p-1} \|_{k} \quad (7) \]

For self-adjoint \( A' \in B(H) \) let \( E_{A'} \) be its associated spectral measure and let \( H_{c} = E_{\pm}(s_{w}(A'), s_{w}(A')) \). For \( m = \text{dim} H \Theta H \), let \( \lambda_{m}(A') \ldots, \lambda_{m}(A') \) be the eigenvalues of \( A' \) in \( H_{c} = H_{c} \), arranged by their non-increasing modulus, with their multiplicities counted. If \( m < \infty \), for all \( n > m \) let \( \lambda_{n}(A') = s_{w}(A') \) if \( s_{w}(A') \in \sigma_{e}(A') \) and \( \lambda_{n}(A') = -s_{w}(A') \) otherwise. Combining the eigen-vectors of \( A' \) in \( H_{c} = H_{c} \) and elements of \( H_{c} = H_{c} \), we can choose a sequence of orthonormal systems \( \left\{ e_{i}^{(n)} \right\}_{i=1}^{\infty} \) such that
\[ \lim_{n \to \infty} \left\| A'e_{i}^{(n)} - \lambda_{i}(A')e_{i}^{(n)} \right\| \to 0 \text{ for all } 1 \leq i \leq k \quad (8) \]

Specially, for \( A' + AX +XA \) it follows from Lemma(2.1), (7) and (8) that
\[ 2^{p-1} \|A\|^{p-1} AX +XA \|A\|^{p-1} \|_{k} \]

\[ \geq 2^{p-1} \limsup_{n \to \infty} \sum_{i=1}^{k} \left| \left( \left( \|A\|^{p-1} (AX +XA) + (AX +XA) \|A\|^{p-1} e_{i}^{(n)} , e_{i}^{(n)} \right) \right) \right| \]

\[ = 2^{p-1} \limsup_{n \to \infty} \sum_{i=1}^{k} \left| \text{Re} \left( (AX +XA)e_{i}^{(n)} , \|A\|^{p-1} e_{i}^{(n)} \right) \right| \]

\[ = 2^{p-1} \limsup_{n \to \infty} \sum_{i=1}^{k} \left| \lambda_{i}(A') \left( \left( \|A\|^{p-1} e_{i}^{(n)} , e_{i}^{(n)} \right) \right) \right|. \quad (9) \]

Spectral representation and Jensen inequality give
\[ \left\langle \|A\|^{p-1} e_{i}^{(n)} , e_{i}^{(n)} \right\rangle = \int_{0}^{\infty} t^{p-1} d \mu_{e_{i}^{(n)}}(t) \geq \left( \int_{0}^{\infty} t^{p-1} d \mu_{e_{i}^{(n)}}(t) \right)^{(p-1)/2} \]

\[ = \left\langle \|A\|^{p-1} e_{i}^{(n)} , e_{i}^{(n)} \right\rangle^{(p-1)/2} = \|Ae_{i}^{(n)}\|^{p-1} \]

\[ \geq \|XAe_{i}^{(n)}\|^{p-1} \geq \text{Re} \left( XAe_{i}^{(n)} , e_{i}^{(n)} \right) \|^{p-1} \]

\[ = 2^{p-1} \left| \left( AX +XA)e_{i}^{(n)} , e_{i}^{(n)} \right) \right|^{p-1} \to 2^{1-p} \| \lambda_{i}(A') \|^{p-1} \]

for \( 1 \leq i \leq k \), and therefore
\[ 2^{p-1} \|A\|^{p-1} AX +XA \|A\|^{p-1} \|_{k} \geq \sum_{i=1}^{k} \left| \lambda_{i}(A') \right|^{p} = \|AX +XA\|^{p} \|_{k}. \quad (10) \]

Having the proof for our special case completed, for arbitrary self-adjoint \( A \) and \( B \) in \( B(H) \) we will consider the following 2×2 self-adjoint operator matrices.
acting on $H \oplus H$. A straightforward calculation gives

$$
C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}
$$

and

$$
\| C \|_{p^{-1}}^p CY +YC \| C \|_{p^{-1}}^{p-1} = \begin{bmatrix} 0 & |A|_{p^{-1}} A X + X B \| B \|_{p^{-1}}^{p-1} \\ (|A|_{p^{-1}} A X + X B \| B \|_{p^{-1}}^{p-1})^* & 0 \end{bmatrix}
$$

and

$$
\| CY +YC \|_p^p = \begin{bmatrix} |(AX +XB)|^p & 0 \\ 0 & |(AX +XB)|^p \end{bmatrix}.
$$

As noted in [129]

$$
s_{2_1/1}(CY +YC) = s_{2_1}(CY +YC) = s_i(AX +XB),
$$

and also

$$
s_{2_1/1}(\| C \|_{p^{-1}}^p CY +YC \| C \|_{p^{-1}}^{p-1}) = s_{2_1}(\| C \|_{p^{-1}}^p CY +YC \| C \|_{p^{-1}}^{p-1}) = s_i(\| A \|_{p^{-1}}^p AX +X B \| B \|_{p^{-1}}^{p-1}).
$$

Applying (10) to self-adjoint $C$ and $Y$ we get

$$
2^{p-2} \| A \|_{p^{-1}}^p AX +XA\| A\|_{p^{-1}}^{p-1} \geq 2^{p-2} \sum_{j=1}^{2k} s_j(\| C \|_{p^{-1}}^p CY +YC \| C \|_{p^{-1}}^{p-1})
$$

$$
\geq 2^{p-2} \sum_{j=1}^{2k} s_j^p(CY +YC)
$$

$$
= \sum_{j=1}^{k} s_j^p(AX +XB) = \| AX + XB \|_p^p \|_k.
$$

Now, by the Ky-Fan dominance property we conclude that this inequality also holds for all unitarily invariant norms $\| \cdot \|_\cdot$.

The preceding theorem can be reformulated as follows.

**Theorem 2.2:** If $X$ and self-adjoint $A$ and $B$ are in $B(H)$, then for all real $0 \leq \alpha \leq 1/3$ then

$$
\| | A | A |_{\alpha^{-1}}^p X + X B | B |_{\alpha^{-1}}^p \|_\cdot \leq 2^{1/\alpha-1} \| X \|_\cdot \| X \|_\cdot \| AX + XB \|_\cdot
$$

for all unitarily invariant norms $\| \cdot \|_\cdot$.

**Proof:** If we denote $C = A|A|_{\alpha^{-1}}$ and $D = D|B|_{\alpha^{-1}}$, then $C$ and $D$ are bounded operators satisfying $\| C \| = |A| \| A \|^{\alpha}$ and $\| D \| = |B| \| B \|^{\alpha}$. Therefore $\| C \| = |C| \| C \|^{\alpha}$ and $\| A \| = |C| \| C \|^{1/\alpha}$, and similarly $\| B \| = |D| \| D \|^{1/\alpha}$. An application of Theorem (2.1) to $C$, $D$ and $1/\alpha$ gives the desired conclusion.
Next, we will show that for $X=1$ the requirement $P \geq 3$ can be relaxed to $P \geq 2$, even for arbitrary bounded operators $A$ and $B$. So, we present the following perturbation inequality for bounded operators.

**Theorem 2.3:** For $A$ and $B$ in $B(H)$ and real $P \geq 2$ we have

$$2^{P-1} \| A - B \|_P \leq 2^{P-1} \| A \|_P - B \|_P \leq P \| A - B \|_P$$

for all unitarily invariant norms $\| \cdot \|_P$.

**Proof:** We will first treat a case with self-adjoint $A$ and $B$ and the Ky-Fan norms $\| \cdot \|_k$. If we define $\lambda_1(A) \geq \lambda_2(A) \geq \cdots$ and $(e_i^{(n)})_{n=1}^\infty$ to be as in the proof of Theorem (2.1), then we similarly get

$$2^{P-1} \| A \|_P - B \|_P \leq 2^{P-2} \| A \|_P - (A - B) \|_P \leq 2^{P-2} \limsup_{n \to \infty} \sum_{i=1}^k \lambda_i(A - B) \| (\langle A \|_P - B \|_P e_i^{(n)}, e_i^{(n)} \rangle) \|_P$$

Jensen inequality shows that

$$\langle A \|_P - B \|_P e_i^{(n)}, e_i^{(n)} \rangle = \langle A \|_P - B \|_P e_i^{(n)}, e_i^{(n)} \rangle \leq \| A \|_P - B \|_P \leq \sum_{i=1}^k \lambda_i(A - B) \| e_i^{(n)} \|_P$$

for $1 \leq i \leq k$ as $n \to \infty$, which together with (11) implies

$$2^{P-1} \| A \|_P - B \|_P \leq (A - B) \|_P \leq 2^{P-1} \| A \|_P - B \|_P \leq \lambda_1(A - B) \|_P$$

for arbitrary $A$ and $B$ in $B(H)$ we consider self-adjoint operator matrices

$$C = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}.$$

A straightforward calculation gives

$$C \| A \|_P - D \| B \|_P \leq 2^{P-1} \| \lambda_1(A - B) \|_P.$$
and we have to note that $A^* \{ A^* \}^{p^{-1}} - B^* \{ B^* \}^{p^{-1}} = (A \{ A \}^{p^{-1}} - B \{ B \}^{p^{-1}})^*$ simply by the fact that both sides of (13) are self-adjoint. According to [18]

We conclude

$$s_{2i-1}(C - D) = s_{2i}(C - D) = s_i(A - B),$$

and also

$$s_{2i-1}(C \{ C \}^{p^{-1}} - D \{ D \}^{p^{-1}}) = s_{2i}(C \{ C \}^{p^{-1}} - D \{ D \}^{p^{-1}})$$

$$= s_i(A \{ A \}^{p^{-1}} - B \{ B \}^{p^{-1}}).$$

As (12) holds for all natural $k$, we get

$$2^{p^{-1}} \| A \{ A \}^{p^{-1}} - B \{ B \}^{p^{-1}} \|_k = \sum_{i=1}^{k} s_i(A \{ A \}^{p^{-1}} - B \{ B \}^{p^{-1}}) = 2^{p^{-1}} \sum_{i=1}^{k} s_i(C \{ C \}^{p^{-1}} - D \{ D \}^{p^{-1}})$$

$$\geq 2^{p^{-1}} \sum_{i=1}^{k} s_i^p(C - D) = \| A - B \|^p \|_k.$$

To conclude the proof, we just have to invoke the Ky-Fan dominance property for bounded operators.

Similarly as it was done in Theorem (2.2), we can give the following reformulation for the previous theorem.

**Theorem 2.4:** For $A$ and $B$ be in $B(H)$ and all real $0 \leq \alpha \leq 1/2$ we have

$$\| A \{ A \}^{\alpha - 1} - B \{ B \}^{\alpha - 1} \|^p \|_{1/\alpha} \leq 2^{1/\alpha - 1} \| A - B \|$$

for all unitarily invariant norms $\| \cdot \|$. 

Constants $2^{p^{-1}}$ and $2^{1/\alpha - 1}$ appearing in previous theorems are sharp, as the simple examples $A=B=X$ and $A = -B = 1$ show. Comparing this with constant 1 obtained in [19], we see that Theorems (2.1)-(2.4) extend the corresponding real and complex numbers inequalities to norm inequalities for self-adjoint and bounded operators, just like in [19] did for the difference of positive operators.

In order to complete the above theorems, we give the following. 

**Theorem 2.5:** For $A$ and $B$ be in $B(H)$, all real $p \geq 1$ and $0 \leq \alpha \leq 1$ there holds

$$\| \alpha A + (1 - \alpha) B \|^p \| \alpha A + (1 - \alpha) B \|^p \|$$

for all unitarily invariant norms $\| \cdot \|$.

**Proof:** Instead of repeating a quite analogous proof, we will just present its essentially different part, saying that for self-adjoint $A$ and $B$ we have

$$\| \alpha A \|^p + (1 - \alpha) \| B \|^p \| \geq \sum_{i=1}^{k} (\alpha \langle A \| e_i \rangle + (1 - \alpha) \langle B \| e_i \rangle \| e_i \rangle$$
\[ \geq \sum_{i=1}^{k} (\alpha |Ae_i,e_i|^p + (1-\alpha) |Be_i,e_i|^p) \]
\[ \geq \left( \sum_{i=1}^{k} |(\alpha A + (1-\alpha)B)e_i,e_i| \right)^p, \quad (14) \]

according to the convexity of the function \( t \to t^p \) for \( p \geq 1 \) and real \( t \). The right-hand side of (14) is approximately \( \|\alpha A + (1-\alpha)B\|_p \) for suitably chosen \( \{e_i\} \), that allows us to end this proof, in which, specifically, and the use of Lemma (2.1) has not been required.

**MEANS INEQUALITIES FOR ELEMENTARY OPERATORS**

We start with the basic Cauchy-Schwarz norm inequality for normal elementary operators. The following theorem extends Theorem 2.5.1 of [8].

**Theorem 3.1:** If \( \sum_{n=1}^{\infty} C_n^* C_n \leq 1, \sum_{n=1}^{\infty} C_n^* C_n \leq 1, \sum_{n=1}^{\infty} D_n^* D_n \leq 1 \) and \( \sum_{n=1}^{\infty} D_n^* D_n \leq 1 \)

for some operator families \( \{C_n\}_{n=1}^{\infty} \) and \( \{D_n\}_{n=1}^{\infty} \), then also \( \sum_{n=1}^{\infty} C_n^* D_n \in C_1 \) whenever \( Y \in C_1 \) for some unitarily invariant norm \( \| \cdot \| \), and moreover

\[ \left\| \sum_{n=1}^{\infty} C_n^* D_n \right\| \leq \| Y \| \quad (15) \]

**Proof:** For arbitrary \( f \) and \( g \) in \( H \) a straightforward calculation gives

\[ \left\| \left( \sum_{n=1}^{\infty} C_n^* Y D_n \right) f, g \right\| \leq \sum_{n=1}^{\infty} \| Y \| \left\| D_n f \right\| \left\| C_n^* g \right\| \]

\[ \leq \| Y \| \left( \sum_{n=1}^{\infty} \left\| D_n f \right\|^2 \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} \left\| C_n^* g \right\|^2 \right)^{\frac{1}{2}} \]

\[ = \| Y \| \left( \sum_{n=1}^{\infty} D_n^* D_n f, f \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} C_n^* C_n g, g \right)^{\frac{1}{2}} \]

\[ = \| Y \| \left( \sum_{n=1}^{\infty} C_n^* C_n \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} D_n^* D_n \right)^{\frac{1}{2}} \left\| f \right\| \left\| g \right\| \]

from which we conclude that

\[ \left\| \sum_{n=1}^{\infty} C_n^* D_n \right\| \leq \| Y \| \quad (16) \]

Therefore, for all \( N = 1,2,... \), for \( Y \in C_1 \) and for all \( W \in B(H) \) we have

\[ \operatorname{tr} \left( \sum_{n=1}^{N} C_n^* Y D_n^* W^* \right) = \operatorname{tr} \left( Y \left( \sum_{n=1}^{N} C_n^* W D_n \right)^* \right) \]
\[ \| Y \| \left\| \sum_{n=1}^{N} C_n WD_n^* \right\| \leq \| Y \| \| W \|, \]

according to (16), from which we deduce that
\[ \left\| \sum_{n=1}^{N} C_n YD_n \right\| \leq \| Y \| \quad (17) \]

If \( Y \in C_\infty \) let \( Y = \sum_{n=1}^{\infty} s_n(Y) \langle \cdot, e_n \rangle f_n \) be a singular value decomposition for some orthonormal systems \( \{ e_n \} \) and \( \{ f_n \} \). For all \( k \geq 2 \) we introduce operators
\[
Z = \sum_{n=1}^{N} (s_n(Y) - s_{n+1}(Y)) \sum_{j=1}^{n} \langle \cdot, e_j \rangle f_j
\]
\[
V = s_k(Y) \sum_{n=1}^{k} \langle \cdot, e_n \rangle f_n + \sum_{n=k+1}^{\infty} s_n(Y) \langle \cdot, e_n \rangle f_n
\]

We see that
\[
Z = \sum_{n=1}^{k-1} \sum_{j=1}^{n} (s_n(Y) - s_{n+1}(Y)) \langle \cdot, e_j \rangle f_j
\]
\[
= \sum_{j=1}^{k} (s_j(Y) - s_k(Y)) \langle \cdot, e_j \rangle f_j
\]
\[
= \sum_{n=1}^{k} s_n(Y) \langle \cdot, e_n \rangle f_n + s_k(Y) \sum_{n=1}^{k} \langle \cdot, e_n \rangle f_n = Y - V
\]

We can also note that \( s_k(V) = \cdots = s_2(V) = s_1(Y) \) due to orthogonality of the systems \( \{ e_n \} \) and \( \{ f_n \} \). That will allow us to conclude that for all Ky Fan \( k \)-norms we have
\[
\left\| \sum_{n=1}^{N} C_n YD_n \right\| \leq \left\| \sum_{n=1}^{N} C_n ZD_n \right\| + \left\| \sum_{n=1}^{N} C_n VD_n \right\|
\]
\[
\leq \| Z \| + k \left\| \sum_{n=1}^{N} C_n ZD_n \right\| \quad (18)
\]
\[
\leq (s_n(Y) - s_{n+1}(Y)) \sum_{j=1}^{n} \left\| \langle \cdot, e_j \rangle f_j \right\| + k \| Y \| \quad (19)
\]
\[
\leq \sum_{n=1}^{k-1} n(s_n(Y) - s_{n+1}(Y)) + k s_k(Y) = \sum_{n=1}^{k} s_n(Y) = \| Y \| \quad (20)
\]

with (18) following from (17) and (19) from (16).

Moreover, if \( Y \) is in \( C_\infty \) then also \( \sum_{n=1}^{\infty} C_n YD_n \in C_\infty \). Indeed, elementary operators \( R_Y = \sum_{n=1}^{N} C_n YD_n \) acting on \( C_\infty^{(K)} \) represent a bounded family, because \( \| R_Y(Y) \| \leq \| Y \| \) for all \( Y \in C_\infty \) by (20). Also, for one dimensional operators \( f \otimes g \) and \( M > N \) we have
\[
\left\| R_M(f \otimes g) - R_N(f \otimes g) \right\| \leq \sum_{n=N+1}^{M} D_n^* f \otimes C_n g
\]
\[ \leq \sum_{n=N+1}^{M} \left\| D_n^* f \right\| \left\| C_n g \right\| \leq \left( \sum_{n=N+1}^{M} C_n C_n^* \right)^{\frac{1}{2}} g \left( \sum_{n=N+1}^{M} D_n^* D_n \right)^{\frac{1}{2}} f \]

which \( \to 0 \) as \( M, N \to \infty \). Therefore \( R_N(Y) \) converge in \( C_{1c}^{(K)} \) for all finite dimensional \( Y \) to a compact operator. By the uniform boundedness principle the same is true for all \( Y \in C_{1c}^{(K)} \), due to its separability. So (15) holds for all Ky Fan k-norms, and we therefore invoke the Ky Fan dominance property to conclude that (15) holds for all unitarily invariant norms, as required.

In the sequel we will refer to a family \( \{ A_n \}_{n=1}^{\infty} \) in \( B(H) \) as square summable if
\[ \sum_{n=1}^{\infty} \| A_n f \|^2 < \infty \]
for all \( f \in H \). Though this means just the weak convergence of \( \sum_{n=1}^{\infty} A_n^* A_n \), an appeal to the resonance principle shows that \( \sum_{n=1}^{\infty} A_n^* A_n \) actually defines a bounded Hilbert space operator, and due to the monotonicity of its partial sums, the convergence is moreover strong. For such families the following Cauchy-Schwarz inequality holds:

**Theorem 3.2:** For a square summable families \( \{ A_n \}_{n=1}^{\infty} \) and \( \{ B_n \}_{n=1}^{\infty} \) of commuting normal operators
\[ \left\| \sum_{n=1}^{\infty} A_n X B_n \right\| \leq \left( \sum_{n=1}^{\infty} A_n^* A_n \right)^{\frac{1}{2}} X \left( \sum_{n=1}^{\infty} B_n^* B_n \right)^{\frac{1}{2}} \]
for all \( X \in B(H) \) and for all unitarily invariant norms \( \| \cdot \| \). If \( C_{1c} \) is separable and \( X \in C_{1c} \), then the left-hand side sum converges in the norm of this ideal.

**Proof:** First, we need a suitable factorization for Hilbert space operators \( A_n \) and \( B_n \). Let
\[ A = (\sum_{n=1}^{\infty} A_n^* A_n)^{\frac{1}{2}} \]
and
\[ B = (\sum_{n=1}^{\infty} B_n^* B_n)^{\frac{1}{2}} \]
and let \( P \) and \( Q \) denote respectively the orthogonal projections on \( \overline{R(A)} \) and \( \overline{R(B)} \). If for a given \( f \in H \) we have that \( Pf = \lim_{k \to \infty} A g_k \) for some sequence \( \{ g_k \} \) in \( H \), then \( \lim_{k \to \infty} A_n g_k \) exists for all \( n \geq 1 \) and does not depend on the chosen sequence. Indeed,
\[ \| A_n g_k - A_n g_l \| \leq \| A(g_k - g_l) \| \to 0 \] as \( k, l \to \infty \), and also \( \| A_n g_k - A_n g_l \| \leq \| A(g_k - g_l) \| \to 0 \) as \( k \to \infty \) whenever \( \lim_{k \to \infty} A h_k = Pf \) for some other sequence \( \{ h_k \} \). Thus we can correctly introduce operators \( C_n, n = 1, 2, \ldots, \) by
\[ C_n f = \lim_{k \to \infty} A_n g_k \]
where \( \{ g_k \} \) is any sequence in \( H \) such that \( \lim_{k \to \infty} A g_k = Pf \). Let us note that due to our definition every \( C_n \) vanishes on \( N(A) \), i.e., \( C_n = C_n P \), and also \( C_n A = AC_n = A_n \).

Moreover, \( \sum_{n=1}^{\infty} C_n C_n^* = P \). Indeed, \( \sum_{n=1}^{\infty} C_n^* C_n A^2 = \sum_{n=1}^{\infty} A_n^* A_n = A^2 \) implies \( \sum_{n=1}^{\infty} C_n C_n^* P = P \), which together with the fact that \( C_n (1 - P) = 0 \) gives the desired conclusion. For all \( m, n = 1, 2, \ldots, C_n \) and \( C_n \) commute on \( R(A^2) \) and \( N(A^2) \), and so also on all of \( H \). Thus \( \{ C_n \}_{n=1}^{\infty} \) is a commuting family of normal contractions which realize the factorizations \( C_n A = AC_n = A_n \), with \( \sum_{n=1}^{\infty} C_n C_n^* = P \), and which commute with the family \( \{ A_n \}_{n=1}^{\infty} \). Similarly
we get a commuting family \( \{ D_n \}_{n=1}^{\infty} \) of normal contractions which also commute with \( \{ B_n \}_{n=1}^{\infty} \) and satisfy \( D_n B = BD_n = B_n \) and \( \sum_{n=1}^{\infty} D_n^* D_n = Q \). One could easily derive the next explicit formula: \( C_n = A_n A_n^* = A_n^* A_n \), where \( A_n^* \) denotes a (densely defined) Moore-Penrose (generalized) inverse for \( A \).

For \( Y = AXB \in C \) (there is nothing to prove in the opposite case), an application of Theorem (3.1) gives

\[
\left\| \sum_{n=1}^{\infty} A_n X B_n \right\| = \left\| \sum_{n=1}^{\infty} C_n Y D_n \right\| \\
\leq \left\| Y \right\| = \left( \sum_{n=1}^{\infty} A_n^* A_n \right)^{1/2} X \left( \sum_{n=1}^{\infty} B_n^* B_n \right)^{1/2} \]  
(22)

which proves the first part of theorem.

Finally, if \( C \) is separable, then for all \( N = 1, 2, \ldots \), an application of the just proven part of theorem combined with the arithmetic-geometric means inequality in [17] gives

\[
\left\| \sum_{n=1}^{\infty} A_n X B_n \right\| = \left\| \sum_{n=1}^{\infty} A_n^* A_n \right\|^{1/2} X \left( \sum_{n=1}^{\infty} B_n^* B_n \right)^{1/2} \\
= \left\| \sum_{n=N}^{\infty} C_n^* C_n \right\|^{1/2} AXB \left( \sum_{n=N}^{\infty} D_n^* D_n \right)^{1/2} \\
= \frac{1}{2} \left( \sum_{n=N}^{\infty} C_n^* C_n \right) AXB + AXB \left( \sum_{n=N}^{\infty} D_n^* D_n \right) \]  
(23)

We see by (22) that \( \{ \sum_{n=N}^{\infty} C_n^* C_n \}_{N=1}^{\infty} \) and \( \{ \sum_{n=N}^{\infty} D_n^* D_n \}_{N=1}^{\infty} \) represent bounded sequences of self-adjoint operators which strongly converge to 0 as \( N \to \infty \). As \( AXB \in C \) which is separable, then the right-hand side of (23) tends to 0 as \( N \to \infty \) by [8]. The conclusion follows.

**Corollary 3.1:** For normal \( A \) and \( B \) in \( B(H) \) and for all real \( r \geq 2 \).

\[
\left\| \frac{AX + XB}{2} \right\| \leq \left( \frac{1 + |A|^r}{2} \right)^{1/2} X \left( \frac{1 + |B|^r}{2} \right)^{1/2} \]  
(24)

as well as

\[
\left\| \frac{X + AXB}{2} \right\| \leq \left( \frac{1 + |A|^r}{2} \right)^{1/2} X \left( \frac{1 + |B|^r}{2} \right)^{1/2} \]  
(25)

for all \( X \in B(H) \) and for all unitarily invariant norms \( \| \cdot \| \).
Proof: \( \{A, I\} \) and \( \{I, B\} \) are families of normal commuting operators, and so for \( r = 2 \) the desired conclusion follows by Theorem (3.2) For \( r > 2 \) the mapping \( t \to t^\frac{r}{2} \) is operator monotone by a well-known Heinz theorem, and therefore this is an operator concave mapping (see [11]). Specifically, \( \frac{1 + |t|^2}{2} \leq \left( \frac{1 + |t|^r}{2} \right)^\frac{2}{r} \), from which we obtain

\[
\left\| \left( \frac{1 + |A|^2}{2} \right)^\frac{r}{2} \left( \frac{1 + |A|^r}{2} \right)^{\frac{2}{r}} \right\| \leq 1
\]

and similarly

\[
\left\| \left( \frac{1 + |B|^2}{2} \right)^\frac{r}{2} \left( \frac{1 + |B|^r}{2} \right)^{\frac{2}{r}} \right\| \leq 1
\]

Therefore

\[
\left\| \left( \frac{1 + |A|^2}{2} \right)^\frac{r}{2} \left( \frac{1 + |A|^r}{2} \right)^{\frac{2}{r}} \right\| \leq \left\| \left( \frac{1 + |B|^2}{2} \right)^\frac{r}{2} \left( \frac{1 + |B|^r}{2} \right)^{\frac{2}{r}} \right\|
\]

which completes the proof.

Corollary 3.2: For normal \( A \) and \( B \) in \( B(H) \) the inequality

\[
\left\| \frac{AX + XB}{2} \right\| \leq \left\| X \right\|^{\frac{r}{2}} \left\| \frac{|A|^r + |X + |B|^r|}{2} \right\|^{\frac{2}{r}}\quad (26)
\]

holds for all real \( r \geq 2 \), for all unitarily invariant norms \( \left\| \right\| \) and for all \( X \in C \).\( \|
\]

Proof: By Corollary (3.1), for all \( t > 0 \),

\[
\left\| \frac{AX + XB}{2} \right\| = t^{-1} \left\| \frac{tAX + Xtb}{2} \right\|
\]

\[
\leq t^{-1} \left\| \left( \frac{1 + |tA|^2}{2} \right)^\frac{r}{2} \left( \frac{1 + |tA|^r}{2} \right)^{\frac{2}{r}} \right\| X \left( \frac{1 + |tB|^2}{2} \right)^\frac{r}{2} \right\|
\]

and therefore

\[
\left\| \frac{AX + XB}{2} \right\| \leq t^{-1} \left\| X \right\|^{\frac{r}{2}} \left\| \left( \frac{1 + |tA|^2}{2} \right)^\frac{r}{2} \left( \frac{1 + |tA|^r}{2} \right)^{\frac{2}{r}} \right\| X \left( \frac{1 + |tB|^2}{2} \right)^\frac{r}{2} \right\|
\]

by [7], because \( \frac{r}{2} < 1 \). Therefore, the arithmetic-geometric mean inequality implies
\[
\left\| \frac{AX + XB}{2} \right\| \leq \frac{1}{2t} \left\| X \right\|^{-\frac{1}{2}} \left( \frac{1 + \left| A^t \right|}{2} \left\| X + \frac{1 + \left| B \right|}{2} \right\| \right)
\leq \frac{1}{2} \left\| X \right\|^{-\frac{1}{2}} \left( t^{-\frac{1}{2}} \left\| X \right\| + t^{\frac{1}{2}} \left\| \frac{A^t X + X \left| B \right|}{2} \right\| \right)^{\frac{1}{2}}
\]

(27)

As the right-hand side equals \(\left\| X \right\|^{-\frac{1}{2}} \left\| \frac{A^t X + X \left| B \right|}{2} \right\|^{\frac{1}{2}}\), which attains its minimum for

\[t = \left\| X \right\|^{-\frac{1}{2}} \left\| \frac{A^t X + X \left| B \right|}{2} \right\|^{-\frac{1}{2}},\]

the conclusion follows.

**Theorem 3.3:** For normal contractions \(A\) and \(B\) the inequality

\[
\left\| (I - A^t A)^{\frac{1}{2}} X (I - B^t B)^{\frac{1}{2}} \right\| \leq \left\| X - AXB \right\|,
\]

holds for all \(X \in B(H)\) and for all unitarily invariant norms \(\left\| \cdot \right\|\).

**Proof:** First, we note that \(s - \lim_{n \to \infty} A^n (I - A^t A)^{\frac{1}{2}} = 0\). Indeed, by a spectral theorem, for every \(f \in H\) there is a positive, finite Borel measure \(\mu\) concentrated on \(D = \{z \in C : |z| \leq 1\}\) such that

\[
\left\| A^n (I - A^t A)^{\frac{1}{2}} f \right\|^2 = \int_D |z|^n (1 - |z|^2) d\mu_f(z)
\]

whence the desired conclusion follows by Lebesgue's dominating convergence theorem. Therefore

\[
w - \lim_{n \to \infty} \left( I - A^t A \right)^{\frac{1}{2}} \left( X - A^t XB^n \right) \left( I - B^t B \right)^{\frac{1}{2}} = (I - A^t A)^{\frac{1}{2}} X (I - B^t B)^{\frac{1}{2}}
\]

So by Theorem (2.3.2) we get

\[
\left\| (I - A^t A)^{\frac{1}{2}} X (I - B^t B)^{\frac{1}{2}} \right\|
\]

\[
= \left\| \lim_{n \to \infty} \left( I - A^t A \right)^{\frac{1}{2}} \left( X - A^t XB^n \right) \left( I - B^t B \right)^{\frac{1}{2}} \right\|
\]

\[
= \left\| \sum_{k=0}^{\infty} \left( I - A^t A \right)^{\frac{k}{2}} A^t (X - AXB) B^t \left( I - B^t B \right)^{\frac{k}{2}} \right\|
\]

\[
\leq \left\| \left( \sum_{k=0}^{\infty} \left| A^k \right|^2 \right)^{\frac{1}{2}} (X - AXB) \left( \sum_{k=0}^{\infty} \left| B^k \right|^2 (I - |B|) \right)^{\frac{1}{2}} \right\|
\]

\[
= \left\| (I - P)(X - AXB)(I - Q) \right\| \leq \left\| X - AXB \right\|,
\]

(29)
where \( P \) and \( Q \) are the orthogonal projections on \( \ker(I - A^*A) \) and \( \ker(I - B^*B) \) respectively. This concludes the proof.

**MAIN RESULTS**

**Theorem 4.1:**

If \( A \) and \( B \) are self-adjoint operators and an arbitrary \( X \) are in \( B(H) \) where \( AX = XB \) then

\[
\| AX + XB \| ^2 \leq 2 \|AX\|^2 + \|XB\|^2
\]

**Proof:**

\[
\| AX + XB \|^2 = \|(AX + XB)(AX + XB)^*\| = \|(AX + XB)(A^*X^* + X^*B^*)\|
\]

\[
= \| AXA^*X^* + AXB^* + XBA^*X^* + XBX^*B^* \|
\]

\[
\leq \| AXA^*X^* + XBX^*B^* \| + \| AXB^* + XBA^*X^* \|
\]

\[
= \| |AX|^2 + |XB|^2 \| + \| |XB|^2 + |AX|^2 \|
\]

\[
= 2 \| |AX|^2 + |XB|^2 \|
\]

**Theorem 4.2:** For self-adjoint normal contraction operators \( A \) and \( B \)

\[
\|(I - A^*)X^2(I - B^*)\| \leq \|X - A^*XB^*\|^2
\]

holds for all \( X \in B(H) \).

**Proof:** Similarly as in the proof of theorem (2.3.5) for \( S = \lim_{n \to \infty} A^*\left(I - \|A^*\|^2\right)^{1/2} = 0 \), therefore

\[
w - \lim_{n \to \infty} \left( I - |A^*|^2 \right)^{1/2} \left( X - A^*XB^* \right) \left( I - |B^*|^2 \right)^{1/2}
\]

\[
= \left( I - |A^*|^2 \right)^{1/2} X \left( I - |B^*|^2 \right)^{1/2}
\]

We get

\[
\left\| \left( I - |A^*|^2 \right) X^2 \left( I - |B^*|^2 \right) \right\|^{1/2}
\]
\[
\begin{align*}
&= \lim_{n \to \infty} \left( I - |A|^2 \right)^{\frac{1}{2}} \left( X - A X B^* \right) \left( I - |B|^2 \right)^{\frac{1}{2}} \\
&= \left( I - |A|^2 \right)^{\frac{1}{2}} \left( \sum_{k=0}^{\infty} A^k (A X B^*) B^{*k} \right) \left( I - |B|^2 \right)^{\frac{1}{2}} \\
&\leq \left( \sum_{k=0}^{\infty} \left( I - |A|^2 \right)^{\frac{1}{2}} \left( X - A X B^* \right) \left( I - |B|^2 \right)^{\frac{1}{2}} \right) \\
&= \left( I - P \right) \left( X - A X B^* \right) \left( I - Q \right) \\
&\leq \left( X - A X B^* \right)
\end{align*}
\]

Hence
\[
\left( I - |A|^2 \right) X^2 \left( I - |B|^2 \right) \leq \left( X - A X B^* \right)^2
\]
REFERENCES


