Reduction of a Multi Criteria Optimization Problem to a Single Criterion Task

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ABSTRACT

This paper is devoted to a problem of a multi criteria optimization of systems. A variant of the linear weighted sum scalarization is described for the reduction of discrete multi criteria optimization problems to a single criterion task. This variant uses the utility concept to transform each of multiple criteria to a uniform numeric form. The reduction is done by defining a generalized criterion as a linear sum of weighted utility values associated with original individual criteria of given optimization problem. The use of utility values instead of actual values of criteria considerably simplifies the aggregation of multiple criteria that in general can be of different types and ranges of values. Created single generalized criterion can be used to determine the best alternative among the given set of alternatives by a single criterion optimization. The detailed algorithm is given that implements all operations to calculate such a criterion and to determine the optimal solution of the problem.

Keywords: Multi criteria optimization, Linear weighted sum scalarization method, Utility concept

INTRODUCTION

In the design of systems of different types we are usually interested in finding a design variant that represents the best choice subject to some restrictions. Typically such a variant is obtained with the use of an appropriate optimization method, on the base of a set of predefined criteria.

With more than one criterion in the search for the best variant of the system, we have a multi criteria optimization problem. Theoretical aspects of solution of such problems are given in works of M. Ehrgott (2005), M. Emmerich and A. Deutz (2006), J. Klapka, P. Pinos, and V. Sevcik (2013), to mention only a few. Examples of practical use of methods of multi criteria optimization are described, in particular, in publications of A. Aleti, et al (2013) and Xin-She Yang (2011). As a rule, these methods require complex algorithms and, in general, they allow to find only approximate solutions.

There are known attempts to solve multi criteria optimization problems with the use of the particle swarm optimization (PSO) method (R. Roy, et al (2012), M. Thamari, et al (2013), M. Nobile, et al (2012)). However, the PSO method itself has serious shortcomings and cannot be considered as a universal optimization technique (S. Kaur, et al (2013)). In particular, it usually works only on real numbers. To apply such a technique to a discrete optimization problem, it is necessary to map the discrete search space to a continuous domain and, after the optimization is performed, to demap the result back into the original discrete space.

There are methods to transform the multi criteria optimization problem into a single criterion problem. One known approach of this kind is epsilon-constrained method, in which one of given original criteria is chosen as an objective function to be optimized, with the remaining criteria considered as constraints (M. Emmerich and A. Deutz (2006)). Another known approach is related to a weighted sum scalarization, where a single objective function is an aggregation of weighted original criteria (M. Ehrgott (2005)).

In this paper, a variant of the linear weighted sum scalarization is described for the reduction of discrete multi criteria optimization problems to a single criterion task. The reduction is done by defining a single generalized criterion as a linear sum of utility values associated with multiple original criteria of the given optimization problem. The use of utility values instead of actual values of criteria considerably simplifies the aggregation of multiple criteria that in general can be of different types with different ranges of their values. The single generalized criterion defined in this way can be used to determine the best design alternative among the given set of alternatives by single criterion optimization. It is assumed further that the set of alternatives and the number of criteria, for all alternatives, are finite.

MULTI CRITERIA OPTIMIZATION

Let $Q_1, Q_2, \ldots, Q_m$ be individual criteria chosen for evaluation of a number of possible design alternatives of some system. With these criteria, $Q = (Q_1, Q_2, \ldots, Q_m)$ is a multidimensional objective function. Assume that the number of alternatives is finite, with alternatives $A_1, A_2, \ldots, A_n$. Formally, the system being optimized can be represented as the following matrix of alternatives:

$$A = \begin{vmatrix} q_{11} \; q_{12} \; \cdots \; q_{1m} \\ q_{21} \; q_{22} \; \cdots \; q_{2m} \\ \cdots \\ q_{n1} \; q_{n2} \; \cdots \; q_{nm} \end{vmatrix} \qquad \ldots\ldots\ldots\ldots (1)$$

In this matrix, rows with numbers 1, 2, …,n represent possible alternatives $A_1, A_2, \ldots, A_n$, and columns with numbers 1, 2, …, m correspond to individual criteria. Each element q_{ij} in the matrix is the value of criterion Q_{ij} in alternative A_{ij}. In general, criteria can be of different types, so that each alternative is represented as a record $A_i = (q_{i1}, q_{i2}, \ldots, q_{im})$. It assumed also that each criteria has values in some range, so that

$$Q_{j,\min} <= Q_j <= Q_{j,\max}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Where $Q_{j,\min}$ and $Q_{j,\max}$ are minimal and maximal values of criterion Q_j for all alternatives, with $j = 1, 2, \ldots, m$.

The problem of multi criteria optimization here is the choice of such alternative A_i, for which we have

$$\max Q_j,\ j = 1, 2, \ldots, m, \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Subject to restrictions (2). However, as was stated in the introduction, to find the best alternative in this case requires the use of complex optimization procedure. We shall show now how the the problem can be reduced to a single criterion task.

REDUCTION OF MULTI CRITERIA OPTIMIZATION PROBLEM TO SINGLE CRITERION TASK

The simplest reduction method is to define a single generalized criterion as the following linear aggregation:

ISSN: 2223-9944, eISSN: 2223-9553

$S = \sum_{j=1}^{m} p_j Q_j$, …………………….. (4)

where p_j is a positive weight of criterion Q_j.

In this case, each alternative A_i can be represented by the sum

$S_i = \sum_{j=1}^{m} p_j q_{ij}$. …………………….. (5)

Unfortunately, this form of the single generalized criterion is not appropriate, since individual criteria can have different types and cannot be summed. In addition, this form does not reflect the extent of satisfaction corresponding to each individual criterion.

To bypass the difficulty of aggregation of multiple criteria of different types and to take into account the aspect of satisfaction corresponding to each individual criterion, we express the single generalized criterion in the following modified form:

$W = \sum_{j=1}^{m} p_j U_j$, …………………... (6)

where p_j has the same meaning as in expression (4), and U_j is utility value of criterion Q_j. Utility value represents the degree of satisfaction of each criterion on a chosen uniform scale. For example, for the criterion "the price of car" we may accept, that utility value for the highest price is zero, while for a low price it is 100. Here zero corresponds to "not acceptable" and 100 to "completely accepted". Utility values between zero and 100 correspond to different levels of "accepted".

To use expression (6), original matrix of multi criteria alternatives (1) must be transformed into the following utility matrix

$$U = \begin{vmatrix} u_{11} & u_{12} & \dots & u_{1m} \\ u_{21} & u_{22} & \dots & u_{2m} \\ & \dots & & \\ u_{n1} & u_{n2} & \dots & u_{nm} \end{vmatrix} \quad \dots\dots\dots\dots\dots (7)$$

In this matrix, each element u_{ij} is utility value of criterion Q_j in alternative A_i. It is important to note that all elements of utility matrix are assumed to have real values. Using this matrix, we can now express the resulting value of each alternative in the form

$V_i = \sum_{j=1}^{m} p_j u_{ij}$ …………………… (8)

Calculating all values V_i and choosing the maximal of them, we determine the optimal of given alternatives. However, to perform cuch calculation it is necessary initially to obtain values of elements of utility matrix (7) and to set weights p_j of the original individual criteria.

MAPPING MATRIX OF ALTERNATIVES TO UTILITY MATRIX

As was stated above, values of different original criteria in matrix (1) have in general different type. In addition, some criteria can be discrete, while others can be continuous. But in any case, as was assumed, their values are restricted according to inequalities (2).

A simple way to transform values of original criteria to their utility values is to use a linear transformation, making utility values lying in some fixed range (for example, in the range of nonnegative numeric values of (0, 100)). Such a transformation can be done according to expression

$u_{ij} = a_j q_{ij} + b_j$, …………………..…… (9)

Where a_j and b_j are scale coefficients corresponding to individual criterion Q_j.

ISSN: 2223-9944, eISSN: 2223-9553
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To determine values of coefficients a_j and b_j in (9), it is necessary to choose limiting values $(u_{ij})_{\min}$ and $(u_{ij})_{\max}$ and associate them with the most preferable and least preferable values of each criterion Q_j, $j = 1, 2, \ldots, m$. Depending on criterion, value $(u_{ij})_{\max}$ can be associated with $(Q_j)_{\min}$ or $(Q_j)_{\max}$.

Suppose that, as an example that values of original criterion Q_j are restricted as

$0.5 <= Q_j <= 1.2$ ………………………..…… (10)

Thus, $(Q_j)_{\min}= 0.5$ and $(Q_j)_{\max}= 1.2$. Assume now, that the designer associates value 0.5 with utility $(u_{ij})_{\min} = 0$ and value 1.2 with utility $(u_{ij})_{\max} = 100$. In this case, coefficients $a_j$and b_j in (9) can be found by solving linear equations

$(u_{ij})_{\min} = 0 \quad = a_j(Q_j)_{\min} + b_j,$

$(Q_j)_{\max} = 1.2 \quad = a_j(Q_j)_{\max} + b_j$ …………. (11)

Solving this system of equations, we will have $a_j= 140$ and $b_j = -70$. Thus, for example, if original criterion has value $q_{ij} = 1.0$ for alternative A_i, then its utility value $u_{ij} = 70$.

Note once again that the association of limiting values $(u_{ij})_{\min}$ and $(u_{ij})_{\max}$ with criterion values $(Q_j)_{\min}$ and $(Q_j)_{\max}$ depends on the criterion. If, for example, criterion Q_j is the price of a car, then it is natural to associate utility value $(u_{ij})_{\min}$ with $(Q_j)_{\max}$ and utility value $(u_{ij})_{\max}$ with $(Q_j)_{\min}$.

If some of the original criteria are not numeric (for examples, colors), then values of such criteria must be initially transformed into appropriate numerical values.

The choice of weights p_j in expression (8) is in general a non-formalized procedure. It can be done by experts on the base of importance of original individual criteria. In such a choice, it is reasonable to ask experts to choose the weights in such a way that they are positive values satisfying, for example, the condition

$\sum_{j=1}^{m} p_{j=1}$ ………………………...………… (12)

THE ALGORITHM

This section presents the algorithm that implements operations described in the previous sections. As was assumed in the introductory section, the number of alternatives n and the number of individual original criteria m are known, with the same number m for each alternative. We assume also that weights p_j, $j = 1, 2, \ldots, m$, are decided on outside the algorithm and are used by algorithm as input data. Values q_{ij} of elements of matrix (1) are also input data. If some original criteria are non-numeric, then it is assumed that they are transformed to a numeric form before being submitted to the algorithm. Thus, the algorithm will deal only with numeric values q_{ij} in (1).

The algorithm is presented as a sequence of numbered steps. These steps are as follows.

[1] Set or input the number of original criteria m and the number of alternatives n.

[2] Input values of original individual criteria q_{ij} for the matrix of alternatives (1).

[3] Input limiting values $(Q_j)_{\min}$ and $(Q_j)_{\max}$ of all criteria Q_j, $j = 1, 2, \ldots, m$.

[4] Input limiting utility values $(u_{ij})_{\min}$ and $(u_{ij})_{\max}$ associated with criteria limits $Q_j)_{\min}$ and $(Q_j)_{\max}$ of all criteria Q_j, $j = 1, 2, \ldots, m$.

[5] Input values of weights p_j,$j = 1, 2, \ldots, m$.

ISSN: 2223-9944, eISSN: 2223-9553
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[6] Compute coefficients a_j and $b_j$for all criteria using the system of linear equations (11).

[7] Consider next alternative A_i, $i = 1, 2, \ldots, n$.

[8] Consider next original criterion Q_j of this alternative, $j = 1, 2, \ldots, m$.

[9] Compute element (utility value) u_{ij} of matrix (7) using coefficients a_j and b_j found at step 6.

[10] If not all criteria are considered, then return to Step 8.

[11] Compute and save the generalized single criterion V_i of the current alternative.

[12] If not all alternatives are considered, then return to Step 7.

[13] Choose $(V_i)_{\max}$ among values $(V_1, V_2, \ldots, V_n)$ as the criterion value representing the optimal alternative A_i, $i \in \{1, 2, \ldots, n\}$.

[14] Output information about the found optimal alternative (such as its index or name and value of its single criterion).

[15] End.

As was mentioned, the algorithm assumes that all elements of matrix of alternatives are numeric. If some criterion originally is not numeric, then it must be mapped to a numeric form. For example, criterion "color" can be mapped to numeric values in this way: white -> 1, black -> 2, red -> 3, and so on.

The main part of the algorithm is implemented as a double loop at steps from 7 to 10. The external loop corresponds to alternatives (rows of matrix of alternatives) and the internal loop to individual criteria (columns of matrix of alternatives).

In addition to information output by the algorithm about the optimal alternative, it can be easily extended to output values of the generalized criterion for all other alternatives. This information can be useful for comparison of alternatives on the base of the generalized criterion.

CONCLUSION

A variant of a linear weighted sum scalarization is described to reduce a discrete multivariate optimization problem to a single criterion task. This variant uses the utility concept to transform multiple criteria of different types to a uniform numeric form and to obtain the single generalized criterion. The detailed algorithm is given that implements all operations necessary to compute such a criterion and to determine the optimal solution of the problem.

ISSN: 2223-9944, eISSN: 2223-9553

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ISSN: 2223-9944, eISSN: 2223-9553