CONVERGENCE PROPERTIES OF THE MODIFIED NUMEROV RELATED BLOCK METHODS

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ABSTRACT

In this paper, we derived the modified 3-point Numerov block method from multistep collocation involving off-step interpolation point $x_{n+1/2}$.The methods were derived for solution of second order initial value problems (IVP's). The single continuous formulation is evaluated at $x_{n+i}, i = 2$ and its first derivative evaluated at $x = x_n$ while its second derivative evaluated at $x = x_{n+i}, i = {}^{1}\!/_{2}$.The combination of the discrete schemes results in a block method which is tested on non- stiff problem at h=0.1 to demonstrate its efficiency.

Keyword: Stiffness ratio, Explicit matching, Non-overlapping Sub-Intervals, Block methods

INTRODUCTION

Very often, the mathematical modeling of many problems be it in physics, chemistry, economics, medicine etc give rise to system of ordinary differential equation. It is well known that initial-value problems of ordinary differential equations often arise in many practical applications too, such as automated control and combustion, chemical reactor, fluid mechanics etc (Aiken, 1985). The traditional methods for solving ODE's generally multistep and Runge-Kutta method methods (Wright, 2002). A linear multistep method (LMM) with continuous coefficients is considered and applied to solve (IVPS's. The well known traditional multistep methods and the hybrid ones can be made continuous through the idea of multistep collocation (Lie & Norsett, 1989; Onumanyi et al., 1994:1999). Following Onymanyi (1994,1999) we identify a continuous formula (cf). The cf is evaluated at some discrete points involving grid and off-grid points along with its first and second derivation, where necessary to obtain multistep discrete formulae for a simultaneous application to the ODE's with initial conditions.

DERIVATION OF THE METHODS

Numerov class of method is suitable for a special class of 2nd (second) order ordinary differential equation of the form:

$$y''(x) = f\big(x, y(x)\big), \qquad a \le x \le b$$
(2.1)

The general linear k-step LMM for (2.1) is given by the difference equation

$$\sum_{j=0}^{k} \alpha_j y_{n+j} - h^2 \sum_{j=0}^{k} \beta_j f_{n+j} = 0$$
(2.2)

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Where $\propto_j$ and β_j are real coefficients $\propto_0, \beta_0$ not both zero with $\propto_k = 1$, we remark that a minimum of three values of y is needed to approximate y'' in (2.1), thus the step number $k \geq 2$, again $2 \leq t \leq k$ or $0 < m \leq k+1$.

Derivation Of Multistep Collocation Methods

For the derivation of the continuous Numerov's class of method, we apply the method carried out by Onumanyi et-al, where a k-step multi-step collocation method with m collocation points was obtained as:

$$\bar{y}(x) = \sum_{j=0}^{t-1} \propto_j(x) y_{n+j} + h^2 \sum_{j=0}^{m-1} \beta_j(x) f(\bar{x}_j, \bar{y}(\bar{x}_j)) \qquad (2.3)$$

with $\bar{y}(x)$ satisfying

$$\bar{y}(x_{n+j}) = y_{n+j},\ j \in \{0, 1, 2, \ldots, t-1\} \qquad (2.4)$$

$$y''(x_j) = f\left(\bar{x}_j, \bar{y}(\bar{x}_j)\right),\ j \in \{0, 1, 2, \ldots, m-1\} \qquad (2.5)$$

Where $\propto_j(x)$ and $\beta_j(x)$ are assumed polynomials of the form

$$\left.\begin{aligned} \propto_j(x) &= \sum_{i=0}^{t+m-1} \propto_{j,i+1} x^i,\ j \in \{0, 1, 2, \ldots, t-1\} \\ h^2\beta_j(x) &= h^2 \sum_{i=0}^{t+m-1} \beta_{j,i+1} x^i,\ j \in \{0, 1, 2, \ldots, m-1\} \end{aligned}\right\} \qquad (2.6)$$

With the following conditions imposed on $\propto_j(x)$ and $\beta_j(x)$.

$$\left.\begin{aligned} \alpha_j(x_{n+i}) &= \delta_{ij}, i, j \in \{0, 1, 2, \ldots, t-1\} \\ h^2\beta_j(x_{n+i}) &= 0, i \in \{0, 1, 2, \ldots, t-1\}, j \in \{0, 1, 2, \ldots, m-1\} \end{aligned}\right\} \qquad (2.7)$$

And

$$\left.\begin{aligned} \alpha_j''(\bar{x}_i) &= 0, i \in \{0, 1, 2, \ldots, t-1\}, j \in \{0, 1, 2, \ldots, m-1\} \\ h^2\beta_j''(\bar{x}_i) &= \delta_{ij}, i, j \in \{0, 1, 2, \ldots, m-1\} \end{aligned}\right\} \qquad (2.8)$$

To obtained $\propto_j(x)$ and $\beta_j(x)$ Onumanyi and Sirisena arrived at a matrix equation of the form

$$DC = I \qquad (2.9)$$

Where I is the identify matrix of dimension (t + m) x (t + m) while D and C are Matrix defined by

$$\begin{pmatrix} 1 & x_n & x_n^{\,2} & x_n^{\,3} & . & . & . & x_n^{\,t+m-1} \\ 1 & x_{n+1} & x^2_{\,n+1} & x^3_{\,n+1} & . & . & . & x_{n+1}^{\,t+m-1} \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & x_{n+t-1} & x^2_{\,n+t-1} & x^3_{\,n+t-1} & . & . & . & x_{n+t-1}^{\,t+m-1} \end{pmatrix}$$

$0 \quad 0 \quad 2 \quad 6\bar{x}_n \quad . \quad . \quad . \quad (t+m-1)(t+m-2)\, x_{n+1}{}^{t+m-3}$

. (2.10)

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$0 \quad 0 \quad 2 \quad 6\bar{x}_{m-1} \quad . \quad . \quad . \quad (t+m-1)(t+m-2)\, x_{m-1}{}^{t+m-3}$

The matrix (2.10) is the multistep collocation matrix of dimension (t + m) x (t + m).

For C, we also defined a matrix of dimension (t + m) x (t + m) whose columns give the continuous coefficients as

$$\begin{pmatrix} \propto_{0,1} & \propto_{1,1} & . & . & . & \propto_{t-1,1} & h^2\beta_{0,1} & h^2\beta_{1,1} & . & . & . & h^2\beta_{m-1,1} \\ \propto_{0,2} & \propto_{1,2} & . & . & . & \propto_{t-1,2} & h^2\beta_{0,2} & h^2\beta_{1,2} & . & . & . & h^2\beta_{m-1,2} \\ . & . & & & & & . & . & & & & . \\ . & & & & & & & & & & & \\ . & . & & & & & . & . & & & & . \\ . & & & & & & & & & & & \\ . & . & & & & & . & . & & & & . \\ . & & & & & & & & & & & \\ \propto_{0,t+m} & \propto_{1,t+m} & . & . & . & \propto_{t-1,t+m} & h^2\beta_{0,t+m} & h^2\beta_{1,t+m} & & & & h^2\beta_{m-1,t+m} \end{pmatrix} \quad (2.11)$$

We defined t as the number of interpolation points while m is the number of collocation points.

From equation (2.9), it follows that

$C=D^{-1}$ (2.12) clearly, (2.12) gives us the continuous coefficients $\alpha_j(x)$ and $\beta_j(x)$.

DERIVATION OF MODIFIED NUMEROV METHOD WITH OFF-STEP POINT $x_{n+1/2}$

Consider the following parameter specifications: $k = 2, t = 3, m = 3. \{x_n, x_{n+\frac{1}{2}}, x_{n+1}\}$ as interpolation points and $\{x_n, x_{n+1}, x_{n+2}\}$ as collocation points, following (2.3) to (2.10), we obtained the D matrix as:

$1 \quad x_n \quad x_n{}^2 \quad x_n{}^3 \quad x_n{}^4 \quad x_n{}^5$

$1 \quad x_{n+\frac{1}{2}} \quad x^2{}_{n+\frac{1}{2}} \quad x^3{}_{n+\frac{1}{2}} \quad x^4{}_{n+\frac{1}{2}} \quad x^5{}_{n+\frac{1}{2}}$

$1 \quad x_{n+1} \quad x^2{}_{n+1} \quad x^3{}_{n+1}. \quad x^4{}_{n+1} \quad x^5{}_{n+1}$. (3.1)

$0 \quad 0 \quad 2 \quad 6x_n \quad 12x_n{}^2 \quad 20x_n{}^3$

$0 \quad 0 \quad 2 \quad 6x_{n+1} \quad 12x^2{}_{n+1} \quad 20x^3{}_{n+1}$

$0 \quad 0 \quad 2 \quad 6x_{n+2} \quad 12x^2{}_{n+2} \quad 20x^3{}_{n+2}$

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The application of section (2.1) to the matrix (3.1) leads to the continuous formulation of the method given by:

$\bar{y}(x) = [(48\xi^5 - 240h\xi^4 + 320h^2\xi^3 - 203h^4\xi + 75h^5)/75h^5]y_n + [(-96\xi^5 + 480h\xi^4 - 640h^2\xi^3 + 256h^4\xi)/75h^5]y_{n+\frac{1}{2}} + [(48\xi^5 - 240h\xi^4 + 320h^2\xi^3 - 53h^4\xi)/75h^5]y_{n+1} + [(-228\xi^5 + 1290h\xi^4 - 2420h^2\xi^3 + 1800h^3\xi^2 - 442h^4\xi)/3600h^3]f_n + [(-204\xi^5 + 870h\xi^4 - 760h^2\xi^3 + 94h^4\xi)/1800h^3]f_{n+1} + [(12\xi^5 - 30h\xi^4 + 20h^2\xi^3 - 2h^4\xi)/720h^3]f_{n+2}$ (3.2)

Where $\xi := x - x_n$

Evaluating (3.2) at $x_n = x_{n+2}$, its 1st derivative at $x = x_n$ and second (2nd) derivative at $x = x_{n+\frac{1}{2}}$ gives the following respective discrete schemes:

$$\left.\begin{aligned} & y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{12}[f_{n+2} + 10f_{n+1} + f_n] \\ & \frac{24}{5}y_{n+1} - \frac{48}{5}y_{n+\frac{1}{2}} + \frac{24}{5}y_n = \frac{h^2}{10}\left[f_{n+1} + 10f_{n+\frac{1}{2}} + f_n\right] \\ & \frac{53}{75}y_{n+1} - \frac{256}{75}y_{n+\frac{1}{2}} + \frac{203}{75}y_n = -hy'_n + \frac{h^2}{3600}[-10f_{n+2} + 188f_{n+1} - 442f_n] \end{aligned}\right\} \quad (3.3)$$

Equation (3.3) is an implicit 3-point Numerov block method with uniform order P = 4 and error constant:

$$C_{p+2} = \left(-\frac{1}{240}, -\frac{1}{3200}, \frac{7}{12000}\right) \quad (3.4)$$

Note. Elimination of $y_{n+\frac{1}{2}}$ between the second and third equations in (3.3) yields

$$\frac{225}{75}y_{n+1} - \frac{225}{75}y_n = 3hy'_n + \frac{h^2}{3600}[30f_{n+2} - 180f_{n+1} + 3840f_{n+\frac{1}{2}} + 1710f_n] \quad (3.5)$$

The explicit matching with non-overlapping sub-intervals can be performed using the block (3.3). However, derivatives must be provided explicitly [15]. Hence, the first derivative of (3.2) at $x = x_{n+2}$ gives

$$hy'_{n+2} = -\frac{203}{75}y_n + \frac{256}{75}y_{n+\frac{1}{2}} - \frac{53}{75}y_{n+1} + \frac{h^2}{3600}[758f_n + 4988f_{n+1} + 1190f_{n+2}] \quad (3.6)$$

Equation (3.6) can now be used for derivation in the computation of the block (3.3).

Many papers have been devoted to the construction and the analysis of Numerov's methods (see e.g a publication in the Journal of the Mathematical Association of Nigeria [15:2002, Vol.29, No.2] see also [16].

CONVERGENCE ANALYSIS

The three integrator proposed in equations (3.3) are put in matrix and for easy analysis the result was normalized t obtain.

$$\begin{pmatrix} 0 & -2 & 1 \\ \frac{-48}{5} & \frac{24}{5} & 0 \\ \frac{-256}{75} & \frac{53}{75} & 0 \end{pmatrix}\begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+2} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \frac{24}{5} \\ 0 & 0 & \frac{203}{75} \end{pmatrix}\begin{pmatrix} y_{n-\frac{1}{2}} \\ y_{n-1} \\ y_n \end{pmatrix} +$$

$$h^2\left\{\begin{pmatrix}0 & \frac{10}{12} & 1\\ \frac{10}{12} & \frac{1}{12} & 0\\ 0 & \frac{188}{2600} & \frac{-10}{2600}\end{pmatrix}\begin{pmatrix}f_{n+\frac{1}{2}}\\ f_{n+1}\\ f_{n+2}\end{pmatrix}+\begin{pmatrix}0 & 0 & \frac{1}{12}\\ 0 & 0 & \frac{1}{12}\\ 0 & 0 & \frac{-442}{2600}\end{pmatrix}\begin{pmatrix}f_{n-\frac{1}{2}}\\ f_{n-1}\\ f_n\end{pmatrix}\right\}$$

Where

$$[A^{(0)}]^{-1}=\begin{bmatrix}0 & \frac{53}{720} & \frac{-1}{2}\\ 0 & \frac{16}{45} & -1\\ 1 & \frac{32}{45} & -2\end{bmatrix}$$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}y_{n+\frac{1}{2}}\\ y_{n+1}\\ y_{n+2}\end{pmatrix}=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}y_{n-\frac{1}{2}}\\ y_{n-1}\\ y_n\end{pmatrix}+h^2\left\{\begin{pmatrix}\frac{53}{864} & \frac{-16859}{561600} & \frac{1}{520}\\ \frac{8}{27} & \frac{-749}{17550} & \frac{1}{260}\\ \frac{16}{27} & \frac{13127}{17550} & \frac{131}{130}\end{pmatrix}\begin{pmatrix}f_{n+\frac{1}{2}}\\ f_{n+1}\\ f_{n+2}\end{pmatrix}+\right.$$

$$\left.\begin{pmatrix}0 & 0 & \frac{3937}{43200}\\ 0 & 0 & \frac{539}{2700}\\ 0 & 0 & \frac{1303}{2700}\end{pmatrix}\begin{pmatrix}f_{n-\frac{1}{2}}\\ f_{n-1}\\ f_n\end{pmatrix}\right\}$$

The first characteristic polynomial of the block method thus would become:

$$\rho(q)=\det\,[qA^{(0)}-A^{(1)}]$$

$$=det\left\{q\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}-\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}\right\}$$

$$=det\left\{\begin{bmatrix}q & 0 & 0\\ 0 & q & 0\\ 0 & 0 & q\end{bmatrix}-\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}\right\}$$

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$$= det\begin{bmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q-1 \end{bmatrix} = q^2(q-1) \Longrightarrow q_1 = q_2 = 0 \; and \;\; q_3 = 1$$

The 1 block 3 point block method is consistent as its order in (3.4) is 4 in which $[4,\ 4,\ 4]^T > 1$. It is also zero stable since $q_1 = q_2 = 0 \; and \; q_3 = 1$.Hence convergent following [8].

Region of Absolute Stability (Ras)

To compute and plot region absolute stability of the block methods, the method of section three are reformulated as general linear methods express as

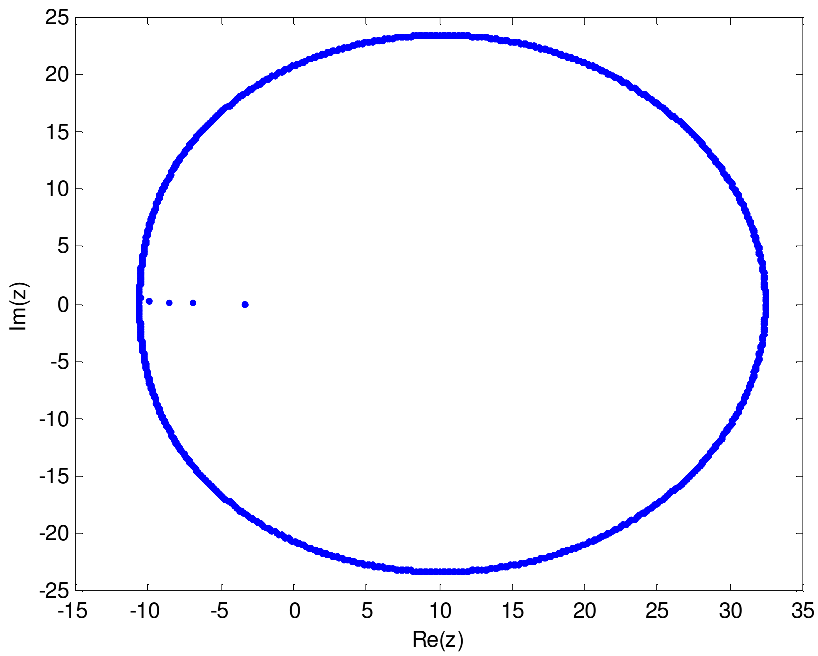
$$\begin{bmatrix} Y \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} A & U \\ B & V \end{bmatrix} \begin{bmatrix} h\,f\,(y) \\ y_{i-1} \end{bmatrix}$$

$$\text{Where } A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{96} & \frac{10}{96} & \frac{1}{96} & 0 \\ \frac{-221}{1272} & 0 & \frac{94}{1272} & \frac{-5}{1272} \\ \frac{1}{12} & 0 & \frac{10}{12} & \frac{1}{12} \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{1}{12} & 0 & \frac{10}{12} & \frac{1}{12} \\ \frac{-221}{1272} & 0 & \frac{94}{1272} & \frac{-5}{1272} \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{-203}{53} \\ 2 & -1 \end{pmatrix} \qquad \text{And } V = \begin{pmatrix} 2 & -1 \\ 0 & \frac{-253}{53} \end{pmatrix}$$

Using a matlab program, the values of the following matrix of A, B, U and V are used to produce the absolute stability region of the block method as shown in fig 4.1



NUMERICAL COMPUTATION AND RESULTS

Consider the second order ODE:

$$y'' + y = 0,\ y(0) = 1,\quad y'(0) = 1,\quad 0 \le x \le 1.2,\quad h = 0.1 \qquad (4.1)$$

$y(x) = Cosx + Sinx$ is the exact solution.

Table 1: Comparism of the theoretical/approximate solutions

x	*Theoretical solution.* $y(x)$	*Numerov block approx. soln.* $y(x)$[15]	*Improved Fatunla Block Approx. soln.* $y(x)$*[16]*	*Modified Numerov Block Approx. soln.* $y(x)$ *(3.3)*
0	1.0	1.0	1.0	1.0
0.1	1.094837582	1.094837379	1.094837566	1.094837583
0.2	1.178735909	1.178735501	1.178735872	1.178735907
0.3	1.250856696	1.250856127	1.250856633	1.250856686
0.4	1.310479336	1.310478608	1.310479212	1.310479314
0.5	1.357008101	1.357007268	1.35700791	1.357008059
0.6	1.389978088	1.389977155	1.389977825	1.389978023
0.7	1.409059875	1.409058895	1.409059502	1.409059777
0.8	1.414062800	1.414061781	1.414062314	1.414062667
0.9	1.404936878	1.404935879	1.404936275	1.404936706
1.0	1.341773291	1.341772316	1.341772537	1.341773076
1.1	1.344803481	1.34480259	1.34480258	1.344803218
1.2	1.29439684	1.294396039	1.294395787	1.294396526

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Table 2: Comparism of the Absolute Errors of the Block Methods

x	*Numerov Block approx. soln. $y(x)$ [15]*	*Improved Fatunla Block Approx. soln. $y(x)$ [16]*	*Modified Numerov Block Approx. soln. $y(x)$ (3.3)*
0	0	0	0
0.1	203×10^{-9}	16×10^{-9}	1×10^{-9}
0.2	408×10^{-9}	37×10^{-9}	2×10^{-9}
0.3	569×10^{-9}	63×10^{-9}	10×10^{-9}
0.4	728×10^{-9}	124×10^{-9}	22×10^{-9}
0.5	833×10^{-9}	191×10^{-9}	42×10^{-9}
0.6	933×10^{-9}	263×10^{-9}	65×10^{-9}
0.7	980×10^{-9}	373×10^{-9}	98×10^{-9}
0.8	1019×10^{-9}	486×10^{-9}	133×10^{-9}
0.9	999×10^{-9}	603×10^{-9}	172×10^{-9}
1.0	975×10^{-9}	754×10^{-9}	215×10^{-9}
1.1	891×10^{-9}	901×10^{-9}	263×10^{-9}
1.2	801×10^{-9}	1053×10^{-9}	314×10^{-9}

CONCLUSION

In this paper, we developed the modified implicit 3-point block Numerov methods of the form (3.3), all the three blocks developed are of order $P = 4$.

These block methods tends to perform better and showed their superiority when compared with other cited works (e.g improved Fatunla's Method,etc.). This indeed is an improvement.

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