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INSTABILITY OF AXIALLY COMPRESSED CCCC THIN RECTANGULAR PLATE USING TAYLOR-MCLAURIN'S SERIES SHAPE FUNCTION ON RITZ METHOD

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ABSTRACT

Compared with conventional structural plates, the pronounced role of instability complicates the behaviour of thin-walled plates. In this study, the stability of in-plane loaded CCCC thin-walled rectangular plate was investigated. The study involved a theoretical formulation based on Taylor-McLaurin series as shape function and implemented through application to Ritz method. In deriving the shape function, Taylor-McLaurin series was truncated at the fifth term, which satisfied the boundary conditions of the plate and resulted to a particular shape function for CCCC plate. The shape function was then substituted into the total potential energy functional, which was subsequently minimized to get the stability equations. Derived Eigen-value solver was used to solve the stability equation for CCCC plates various aspect ratios to get the buckling loads. The buckling loads from this study were compared with those of earlier researchers and the average percentage difference recorded for CCCC plate was 3.54%. This difference shows that the shape function derived from Taylor-McLaurin series has rapid convergence and is a very good approximation of exact displacement function of the deformed thin-walled rectangular plate under in-plane loading.

Keywords: Instability; Thin-walled Plates; Taylor-Mclaurin Series; Total Potential Energy Functional; Eigen-Value Solver; Critical Buckling Load

INTRODUCTION

The critical buckling load of a rectangular plate is usually given as $(\mathrm{Nx})_{\mathrm{cr}} = \frac{\mathrm{H\,D}\,\pi^2}{\mathrm{b}^2}$. In this expression, H is the coefficient, D is the flexural rigidity of the plate and b is the length of the loaded edge. The value of H is dependent on the aspect ratio a/b. Earlier scholars have used Euler (equilibrium) approach, energy approach and numerical approach in analyzing plates clamped at all the four edges (CCCC – plates). (Levy, 1942) used an infinite series as shape function in equilibrium approach and obtained H for a square thin rectangular plate for first buckling mode to be 10.07. (Timoshenko, 1936), used trigonometric series as shape function in energy approach and got H for a square thin rectangular plate to be 9.362. Iyengar (1988), used the variation of total potential energy method, and assumed a shape function, W= A (1- Cos2πR) (1- Cos2πQ), where $R = {}^{x}/_{a}\ and\ Q = {}^{y}/_{b}$, to analyze the plate. He obtained H for a square thin rectangular plate to be 10.667. The difference between these solutions from various approaches and the use of different shape functions calls for further study. This is to ascertain the solution that could converge close to the exact solution. In the use of trigonometric series, single Fourier series or double Fourier series could be used. However, their convergence is slow. This is why Krylov (1949) proposed an efficient method for sharpening the convergence of the Fourier series. Kantorovich and Krylov (1954) also

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presented solutions of plate by approximate methods of higher analysis. They made use of Fourier series in these approximate methods. Many other researchers like Nadai (1925), Timoshenko and Woinowsky-Krieger (1959), Iyenger (1988), Ye (1994), Ugural (1999), Ventsel and Krauthammer (2001) and Eccher, Rasmussen and Zandonini (2007) also used Fourier series in their work.

It is of important note that previous scholars have extensively used trigonometric series in energy approach. Use of Taylor series has attracted very little attention and has not been used to estimate shape functions for use in energy approach for analyzing CCCC plate. Thus, this paper used Taylor-Mclaurin's series to solve the problem of a CCCC plate and subject to in-plane load in one axis (X - axis) of the principal plane (figure 1.).

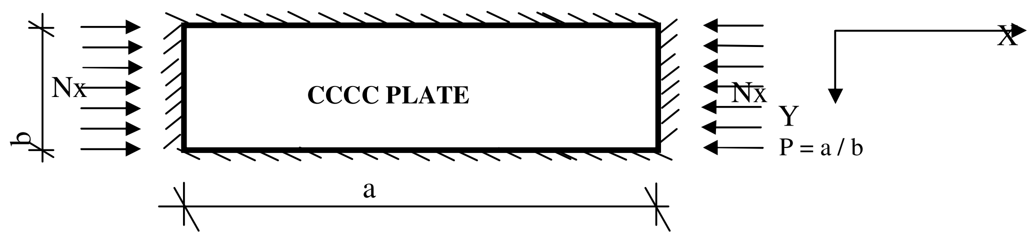

Figure 1: schematic representation of in-plane loaded CCCC plate

TOTAL POTENTIAL ENERGY FUNTIONAL FOR THIN PLATE BUCKLING

Ibearugbulem (2011) derived the total potential energy functional for a rectangular thin isotropic plate subjected to in-plane load in x-direction as follows:

$$\prod_x = \frac{Db}{2a^3}\iint\left[\left(w''^{R}\right)^2 + \frac{a^4}{b^4}\left(w''^{Q}\right)^2 + 2\frac{a^2}{b^2}\left(w''^{RQ}\right)^2 + 2\mu\frac{a^2}{b^2}w''^{R}.w''^{Q} - 2\mu\frac{a^2}{b^2}\left(w''^{RQ}\right)^2\right]\partial R\,\partial Q - \frac{bN_x}{2a}\iint\left(w'^{R}\right)^2\partial R\,\partial Q. \qquad (1)$$

Where "a" and "b" are plate dimensions in x and y directions. μ is Poisson's ratio. Nx is the in-plane load in x direction. D is flexural rigidity. W is the shape function. $\prod_x$ = total potential energy functional along x axis; $R = \frac{x}{a}$; $Q = \frac{y}{b}$;

$0 \le R \le 1;\ 0 \le Q \le 1$ (R and Q are dimensionless quantities).

$\frac{\partial w}{\partial x} = \frac{1}{a}\frac{\partial w}{\partial R} = \frac{1}{a}w'^{R}$; $\frac{\partial^2 w}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 w}{\partial R^2} = \frac{1}{a^2}w''^{R}$; $\frac{\partial w}{\partial y} = \frac{1}{b}\frac{\partial w}{\partial Q} = \frac{1}{b}w'^{Q}$

$\frac{\partial^2 w}{\partial y^2} = \frac{1}{b^2}\frac{\partial^2 w}{\partial Q^2} = \frac{1}{b^2}w''^{Q}$; $\frac{\partial^2 w}{\partial x\partial y} = \frac{1}{ab}\frac{\partial^2 w}{\partial R\partial Q} = \frac{1}{ab}w''^{RQ}$.

If the chosen shape functions is a good approximation of the exact shape function, then $\int_0^1\int_0^1 w''^{R}.\,w''^{Q}\,\partial R\partial Q - \int_0^1\int_0^1 (w''^{RQ})^2\,\partial R\partial Q \approx 0$

(Ventsel & Krauthammer, 2001), (Ibearugbulem, 2011). Thus, equation (1) becomes:

$$\prod_x = \frac{D}{2b^2}\int_0^1\int_0^1\left[\frac{b^3}{a^3}\left(w''^{R}\right)^2 + \frac{a}{b}\left(w''^{Q}\right)^2 + \frac{2b}{a}\left(w''^{RQ}\right)^2\right]\partial R\,\partial Q - \frac{bN_x}{2a}\int_0^1\int_0^1 (w'^{R})^2\,\partial R\,\partial Q. \qquad (2)$$

SHAPE FUNCTION FROM TAYLOR-MCLAURIN'S SERIES

Ibearugbulem (2011) assumed the shape function, w to be continuous and differentiable. He expanded it in Taylor-Mclaurin series and got:

$$w = w(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{F^{(m)}(x_0).F^{(n)}(y_0)}{m!\,n!}(x - x_0)^m.(y - y_0)^n \qquad (3)$$

Where $F^{(m)}(x_0)$ is the mth partial derivative of the function w with respect to x and $F^{(n)}(y_0)$ is the nth partial derivative of the function w with respect to y. m! and n! are factorials of m and n respectively. x_0 and y_0 are the points at the origin. He took the origin to be zero. After some modification of equation (1) and noting that x = a . R and y = b . Q, he truncated the infinite series at m = n = 4 and got:

$$w = \sum_{m=0}^{4}\sum_{n=0}^{4} J_m\, K_n\, R^m.Q^n \qquad (4)$$

Where $J_m = \dfrac{F^{(m)}(0) * a^m}{m!}$ and $K_n = \dfrac{F^{(n)}(0) * b^n}{n!}$

The boundary conditions for CCCC plate are

$$w(R = 0) = w'^R(R = 0) = 0 \qquad (5)$$

$$w(R = 1) = w'^R(R = 1) = 0 \qquad (6)$$

$$w(Q = 0) = w'^Q(Q = 0) = 0 \qquad (7)$$

$$w(Q = 1) = w'^Q(Q = 1) = 0 \qquad (8)$$

Substituting equations (5) and (7) into equation (4) gave:

$J_0 = 0$; $J_1 = 0$; $K_0 = 0$; $K_1 = 0$

Also, substituting equation (6) into equation (4) and solving the resulting two simultaneous equations gave:

$J_2 = J_4; J_3 = -2J_4$

Similarly, substituting equation (8) into equation (4) and solving the resulting two simultaneous equations gave:

$K_2 = K_4; \; K_3 = -2K_4$

Substituting the values of J_0, J_1, J_2, J_3, J_4, K_0, K_1, K_2, K_3 and K_4 into equation (4) gave

$w = (R^2 - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4)J_4K_4$. That is

$$w = A(R^2 - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4) \qquad (9)$$

Where $A = J_4K_4$

APPLICATION OF RITZ METHOD

Partial derivatives of equation (9) with respect to either R or Q or both gave the following equation:

$$w'^R = A(2R - 6R^2 + 4R^3)(Q^2 - 2Q^3 + Q^4) \qquad (10)$$

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$$w''^{R} = A(2 - 12R + 12R^2)\,(Q^2 - 2Q^3 + Q^4) \qquad (11)$$

$$w'^{Q} = A(R^2 - 2R^3 + R^4)\,(2Q - 6Q^2 + 4Q^3) \qquad (12)$$

$$w''^{Q} = A(R^2 - 2R^3 + R^4)\,(2 - 12Q + 12Q^2) \qquad (13)$$

$$w''^{RQ} = A(2R - 6R^2 + 4R^3)\,(2Q - 6Q^2 + 4Q^3) \qquad (14)$$

Integrating the square of these five equations partially with respect to R and Q in a closed domain respectively gave:

$$\int_0^1\int_0^1 \left(w'^{R}\right)^2 \partial R\,\partial Q = 0.00003A^2 \qquad (15)$$

$$\int_0^1\int_0^1 \left(w''^{R}\right)^2 \partial R\,\partial Q = 0.00127A^2 \qquad (16)$$

$$\int_0^1\int_0^1 \left(w'^{Q}\right)^2 \partial R\,\partial Q = 0.00003A^2 \qquad (17)$$

$$\int_0^1\int_0^1 \left(w''^{Q}\right)^2 \partial R\,\partial Q = 0.00127A^2 \qquad (18)$$

$$\int_0^1\int_0^1 \left(w''^{RQ}\right)^2 \partial R\,\partial Q = 0.00036A^2 \qquad (19)$$

Substituting equations (15), (16), (18) and (19) into equation (2) gave:

$$\prod x = \frac{DA^2}{2b^2}\left(\frac{0.00127}{P^3} + \frac{0.00073}{P} + 0.00127P\right) - \frac{N_x A^2}{2P}(0.00003) \quad (20)$$

$$\text{Where the aspect ratio, } P = \frac{a}{b} \qquad (21)$$

Minimizing equation (20) and solving the resulting eigen-value equation gave:

$$N_x = \left(\frac{4.255}{P^2} + 2.428 + 4.255P^2\right)\frac{\pi^2 D}{b^2} \qquad (22)$$

That is to say

$$N_x = \frac{D\pi^2}{b^2}.H \qquad (23)$$

$$\text{wher } H = \left(\frac{4.255}{P^2} + 2.428 + 4.255P^2\right) \qquad (24)$$

RESULTS AND DISCUSSION

The H values from this paper and those from Iyengar (1988) were presented and compared on table 1.

Exact solution from Iyengar (1988) was

$$(N_x)_{cr} = \frac{D\,\pi^2}{b^2}\left(\frac{M^2}{p^2} + \frac{P^2}{M^2} + 2\right)$$

Where M is the buckling mode. In this case it is the first mode. That is M = 1, then we have

$$(N_x)_{cr} = \frac{D\,\pi^2}{b^2}\left(\frac{1}{P^2} + P^2 + 2\right)$$

The values of H for different aspect ratios are shown on table 1. Here, average percentage difference between this study and Iyenger (in this case solution from trigonometric function, table 1) is 3.538%. This is an upper bound approximation. It would also be noticed that the closeness of the two solutions improves as the aspect ratio increases from 0.1 to 1.0, given the corresponding percentage difference from 5.528% to 1.978%. This meant that the solution from this present study was a close approximate of the exact solution if it is assumed that solution from trigonometric functions are close to exact solution. Hence, the assumed deflection function was close to the exact shape function. Furthermore, (Levy,1942), used an infinite series to get H for CCCC plate with aspect ratio of 1.0 as 10.07. However, H from this study is 10.878. This is an upper bound solution in comparison with Levy's solution. The average percentage difference was 8.024%. It would be nice to know that Iyenger's solution differs from Levy's by 5.929%. These differences are quite acceptable in statistics as being close. It went further to affirm the good approximation of the shape function using finite power series. However, it is not yet certain which of the solutions (Iyenger or Levy) is closer to the exact solution.

Table 1. H values for different aspect ratios for CCCC Thin plate buckling

ASPECT RATIO, P = a/b	*H from IYEENGAR (1988)*	*H from PRESENT STUDY*	*PERCENTAGE DIFFERENCE*
0.1	402.707	424.970	5.528
0.2	102.827	108.222	5.247
0.3	47.471	49.753	4.806
0.4	28.307	29.510	4.251
0.5	19.667	20.384	3.647
0.6	15.218	15.685	3.069
0.7	12.790	13.121	2.583
0.8	11.477	11.734	2.235
0.9	10.845	11.066	2.038
1	10.667	10.878	1.978

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