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COMPARATIVE STUDY ON USE TRIANGULAR AND RECTANGULAR FINITE ELEMENTS IN ANALYSIS OF DEEP BEAM

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ABSTRACT

The classification of computational solid mechanics (CSM) for static analysis is based on discretization method by which the continuum mathematical model is discretized in space. Being one of the numerical methods, Finite Element Method (FEM) has the advantage of retaining the physical characteristics of the problem throughout the process of its solution. The domain of interest is represented as an assembly of finite elements and the approximating functions are determined in terms of nodal values of a physical field. Several approaches can be used to transform the physical formulation of the problem to its finite element discrete analogue. For all the methods available, the method used in discretizing the continuum contributes substantially to the degree of accuracy of the results. In this study, triangular and rectangular finite element models were used in a comparative study to determine the nodal displacements in the numerical analysis of a deep beam. The results of the analysis show that the use of rectangular elements yielded results that are closer to the exact solutions than the triangular element model.

Keywords: Comparative Study, Deep Beam, Finite Element Method, Numerical Method, Rectangular Elements, Triangular Elements.

INTRODUCTION

The structural behaviour of deep beam is not completely understood. The behaviour of deep beams differs from those of shallow beams due to characteristic small ratio between shear span and depth (Melvin, & Joost, 1995). In current design practice, structural analysis for reinforced concrete frames is generally based on the assumption that plane sections remain plane after bending and the material is homogeneous and elastic (Enem *et al.,* 2012; Writer *et al.,* 1989). Therefore, linear elastic methods of analysis are normally adopted for the analysis of simple reinforced concrete beams and frames to obtain the member forces and bending moments that will enable the design and detailing of the sections to be carried out, despite the fact that reinforced concrete is not a homogenous and elastic material (Structural Use of Concrete, British Standard Institution, 1985).

However, the elementary theory of bending for simple beams may not be applicable to deep beams even under the linear elastic assumption. There are several methods available for the analysis of deep beams that are either simply supported or continuous. Considerable number have been used, neither of which is entirely satisfactory (Khalaf, 2007). But finite element method (FEM) has proved to be a more versatile tool compared with other methods of computational solid mechanics. In applying finite element method to a problem, it is customary to discretize the continuum into finite elements. By using many elements, it virtually approximates any continuum with complex boundary and loading conditions to such as degree that an accurate analysis can be carried out. Hence, utmost care and attention should be paid to the choice of size of elements and manner of discretiztion. Owing to the above fact, two different finite element models were considered in this study. Comparative study was carried out on the use of triangular and rectangular discretizations in the analysis of deep beams and their complete mathematical formulation can be found in finite element texts (Zienkiewicz *et al.,* 2000; Klus–Jurgen, 1990).

CONCEPT OF FINITE ELEMENT METHOD

The finite element method can be regarded as an extension of the displacement method for beams to two and three dimensional continuum problems, such as plates, shells and solid bodies. The actual continuum is replaced by an equivalent idealized structure composed of discretized elements connected together at a finite number of nodes. By assuming displacement fields of stress patterns within an element it is possible to derive a stiffness matrix relating the nodal forces to the structure, which is the assemblage of all the elements, is then obtained by combining the individual stiffness matrices of all the elements in the proper manner. If conditions of equilibrium are applied at every node of the idealized structure, a set of simultaneous equations can be formed, the solution of which gives all the nodal displacements, which in turn are used to calculate all the internal stresses (Ghali *et al.,* 1972).

DESCRIPTION OF THE BEAM (MODEL)

The geometry of the beam used in this comparative analysis is shown in figure 1. The beam is modeled as a continuous deep beam with each span equal to 6m and a total depth of 1.8m. The beam is supported at edges and middle. Two–point loading is employed in the analysis.

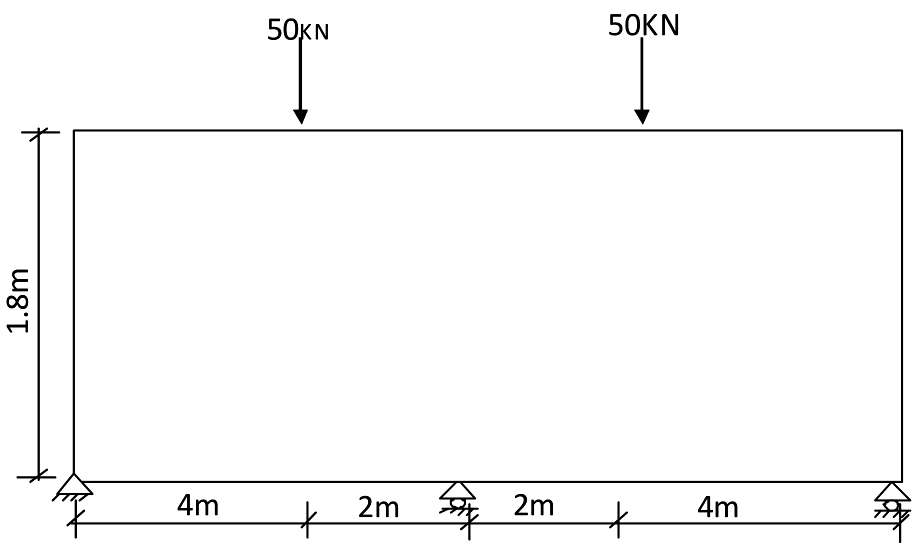

Figure 1. Model for continuous deep beam under two–point loading.

DISCRETIZATION OF THE DOMAIN

Discretization of a global domain into sub-domain or continuum is the first step in any analysis. And this is the process of subdividing the physical body or structure that is being analyzed into finite elements. Figure 2 represents the discretized domain which is represented by a discrete of nodal points connected by elements.

These finite elements are discrete representation of a continuous physical system. However, the discretization is essentially an exercise of engineering judgment and the resulting finite element mesh is defined by the type of discretization method that is used (Triangular or Rectangular). From figure 2,it is obvious that both models gave the same number of nodes but Triangular discretization generates more elements (Edward, 1992).

DISPLACEMENT FUNCTION

The choice of suitable displacement function is the most important part of the whole procedure. A good displacement function will lead to an element of high accuracy and with converging characteristics, and conversely a wrongly chosen displacement function yielding poor or non–converging results and at times even worse, solutions which converge to an incorrect answer (Chung *et al.,* 1979). Figure (2) also shows the arrangement of the displacement for both methods. In this research, only displacement models were considered.

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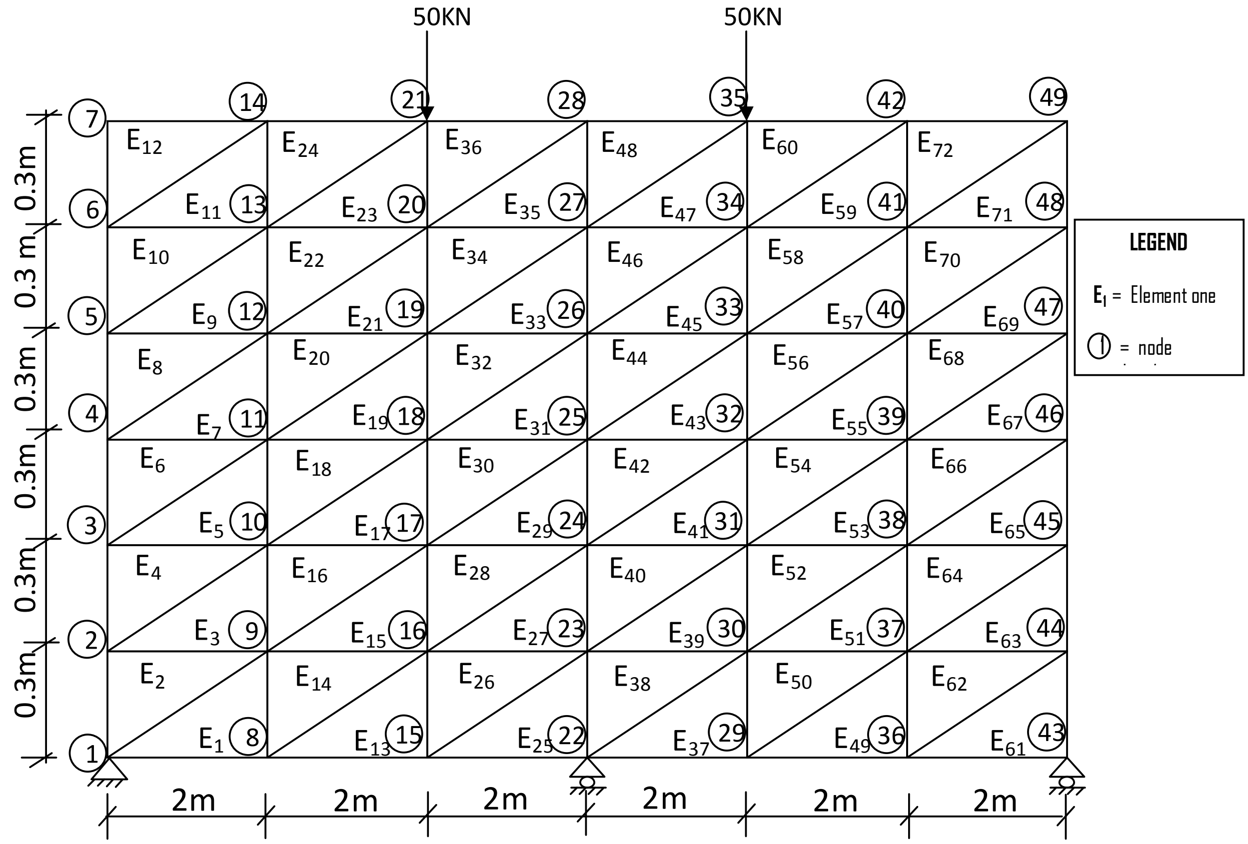
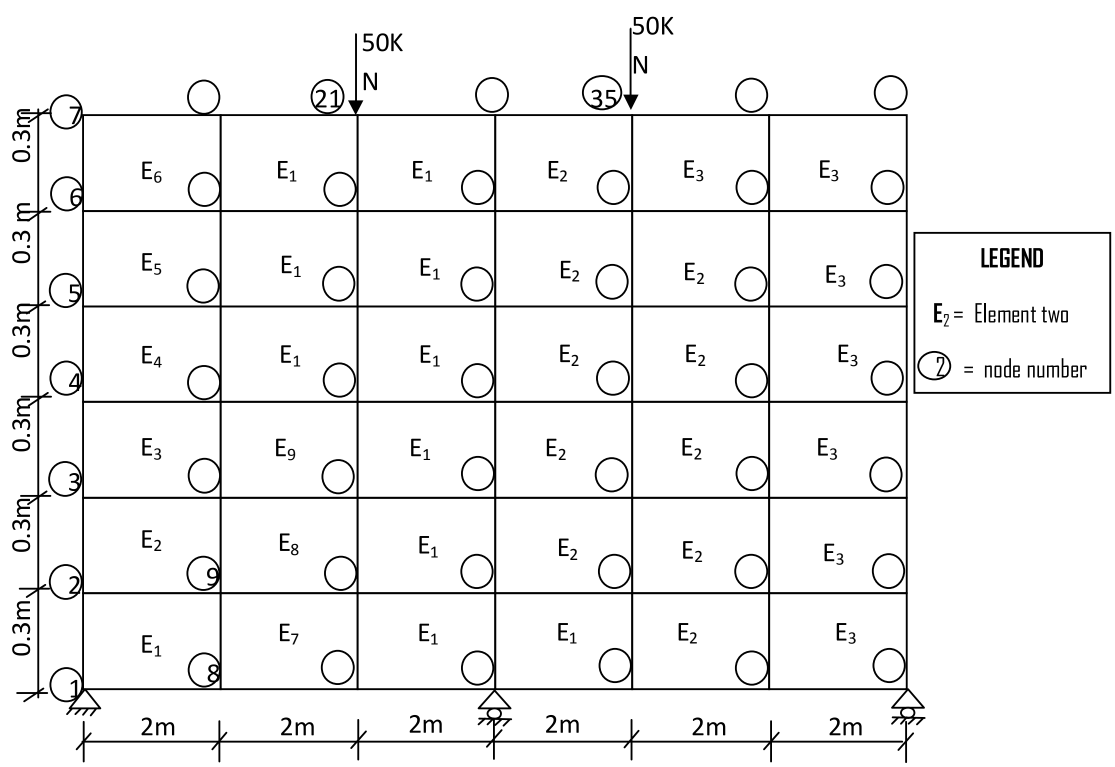


Figure 2b. Rectangular discretization.



FORMULATION OF STIFFNESS MATRIX (K)

Stiffness matrix can be essentially classified into element matrix (element analysis) and global matrix (system analysis). Basically, the elements analysis is the study of the individual finite elements into which the structure is idealized. It involves the determination of all the element stiffness matrices whereas the global matrix involves the formulation of global stiffness matrix. This process is achieved by superimposing the individual element stiffness matrices in a systematic manner. The procedure and mathematical formulas for generating stiffness matrix for the two idealizations are illustrated below.

TRIANGULAR STIFFNESS MATRIX

In constructing an element stiffness matrix (K), the material stiffness (D) is required to relate stresses { σ } to strains { ε }, that is

$$\{\sigma\} = [D]\{\varepsilon\} \qquad - \quad - \quad - \quad - \quad - \qquad (1)$$

Where,

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \; and \; \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \Upsilon_{xy} \end{Bmatrix}$$

For a linear elastic isotropic material, in a plane stress state,

$$[\mathbf{D}] = \frac{\mathbf{E}}{1 - v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \qquad - \quad - \quad - \quad - \qquad (2)$$

The stresses caused by the element nodal displacements are related by the following relation;

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\mathbf{D}]\{\varepsilon\} = [\mathbf{D}][\mathbf{B}]\{\delta\} \qquad - \quad - \qquad (3)$$

From virtual work principle, it can be estimated that the element nodal forces induced by the nodal displacements are given by;

$$F = \iint [B]^T [D][B] t\,dx\,dy * \{\delta\} \qquad - \quad - \qquad (4)$$

Where t denotes the thickness of the element and hence the stiffness matrix of the element is;

$$[K] = \iint [B]^T [D][B] t\,dx\,dy \qquad - \quad - \quad - \qquad (5)$$

Since all the terms are constant, the integral ∬dxdy over the whole area of the element is just its area Δ. Hence

$$[F] = [B]^T [D][B] t\Delta * (\delta) = [K]\{\delta\} \qquad - \quad - \qquad (6)$$

Where,

$$[K] = [B]^T [D][B] * t * \Delta \qquad - \quad - \quad - \qquad (7)$$

RECTANGULAR STIFFNESS MATRIX

It is more convenient to use isoparametric quadrilateral element which must be formulated through the use of natural co–ordinates (ξ,η) and co–ordinate transformation techniques (Tirrupathi *et al.*,2007).

Numerical integration (by Gaussian Quadrature)

Consider the function below (n–point approximation)

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$$I = \int_{-1}^{1} f\,(\xi) = W_1\, f\,(\xi) + W_2\, f\,(\xi) + \ldots + W_n f\,(\xi n) \quad - \quad -- \qquad (8)$$

Where, W_1 , W_2… and W_n are the weights and ξ_1, ξ_2…., and ξ_n are the sampling points or Gauss points. The ideal behind Guassian quadratra is to select the n Gauss points and n weights such that equation 8 provides an exact answer for polynomials $f\,(\xi)$ of as large a degree as possible (Kong, 2000).

It is concluded that n–point Gaussian quadrature will provide an exact answer if f is a polynomial of order (2n–1) or less. Table 1 gives the values of W_1 and ξ_1 for Gauss quadrature formula of order n = 1 to n = 6.

Table 1. Gauss point and weight for Gaussian Quadrature

Number of point n	Locations ξi	Weights, W_i
1	0.0	2.0
2	± 0.577350262	1.0
3	± 0.774966692	0.5555555556
	0.0	0.8888888889
4	± 0.8611363116	0.3478548451
	± 0.3399810436	0.6521451549
5	± 0.9961798459	0.2369268851
	± 0.5384693101	0.4786286705
	0.0	0.5688888889
6	± 0.912463142	0.1713244924
	± 0.6612093865	0.360765730
	± 0.2386191861	0.4679139346

$$\int_{-1}^{1} f(\xi)d\xi = \sum_{i=1}^{n} W_i f(\xi)$$

NB: The large number of digits given in table 1 should be used in the calculations for accuracy.

Two–Dimensional Integral

NB: The large number of digits given in table 1 should be used in the calculations for accuracy.

Two–Dimensional Integral

The extension of Gaussian quatrature to two–dimensional integrals of the form

$$I = \int_{-1}^{1}\int_{-1}^{1} f\,(\xi,\zeta)\,d\xi\, d\,\zeta \qquad - \quad - \quad - \qquad (9)$$

Follows readily, since $I = [\Sigma_{\downarrow}(i=1)^{\uparrow}n \equiv \square W_{\downarrow}i\, f(\xi,\text{"}(\text{"})\ \square]d($ ------10

$$I = \Sigma_{\downarrow}(i=1)^{\uparrow}n \equiv \sqcup W\,[W_{\downarrow}i\, f(\xi,\text{"}(\text{"})\,]\sqcup d(\quad -------11$$

$$OR \qquad I = \sum_{i=1}^{n}\sum_{j=1}^{n} W_i W_i f(\xi, () \qquad ---------- \ 12$$

Stiffness Integration

To illustrate the use of equation 12, consider the element stiffness for a quadrilateral element.

$$K^e = t_e \int_{-1}^{1}\int_{-1}^{1} B^T \,(\, B \det J \, d\xi \, d\,(\qquad - \ - \ - \ - \ - \qquad (13)$$

Where B and det J are function of ξ an dη. Note that the integral actually consists of the integral of each element in an (8 x 8) matrix. However, using the fact that K^e is symmetric we do not need to integrate elements below the main diagonal.

Let ϕ represent the ijth element in the integral

$$((\xi, "(") = t_{\downarrow}e \,(B^{\uparrow}T \, B \det j)ij \qquad - \ - \ - \ - \ - \ - \qquad (14)$$

Then, if we use a 2 x 2 rule, we get

$$K_{\downarrow}ij = W_{\downarrow}1^{\uparrow}2 \,((\xi_{\downarrow}1, (_{\downarrow}1\,) + W_{\downarrow}1 \, W_{\downarrow}2 \,((\xi_{\downarrow}1, (_{\downarrow}1\,) + W_{\downarrow}2 \, W_{\downarrow}1 \,((\xi_{\downarrow}1, (_{\downarrow}1\,) + W_{\downarrow}2^{\uparrow}2 \,(\xi_{\downarrow}2, (_{\downarrow}2\,)$$

---- (15)

For four – node quadrilateral,

$$W_{\downarrow}1 = W_{\downarrow}2 = 1.0, \qquad \xi_{\downarrow}1 = (_{\downarrow}1 = -0.5773502692 \ and \ \xi_{\downarrow}2 = (_{\downarrow}2 = +0.5773502692$$

The Gauss points for the two–point rate used above are shown in figure 3.

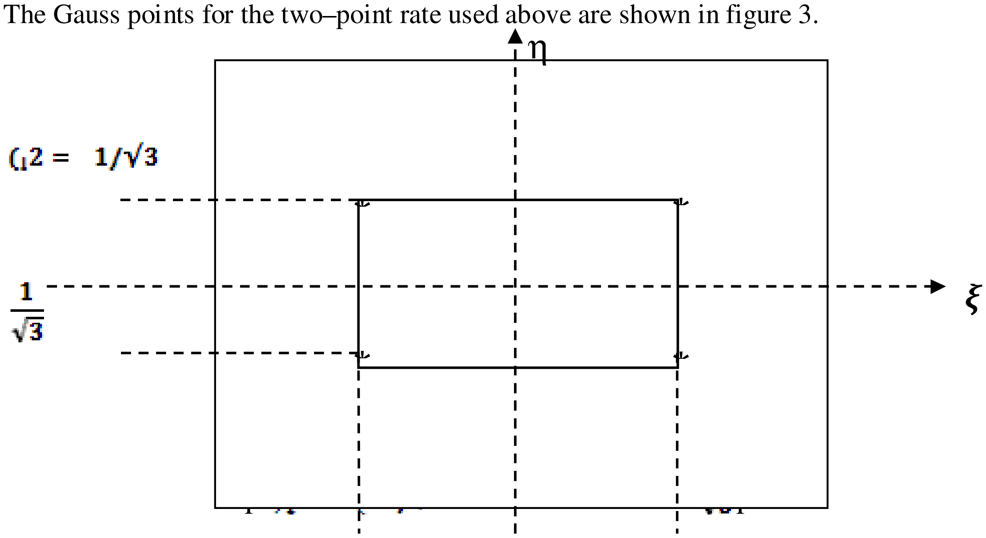

Figure 3: Gaussian quadrature in two dimensions using 2 x 2 rules.

$$\int_{\downarrow}(-1)^{\uparrow}1 \int_{\downarrow}(-1)^{\uparrow}1 \, f\,(\xi, "(")\, d\xi \, d" (" = W_{\downarrow}1^{\uparrow}2 \, f(\xi_{\downarrow}1, (_{\downarrow}1\,) + W_{\downarrow}1 \, W_{\downarrow}2 \, f(\xi_{\downarrow}2, (_{\downarrow}1\,) + W_{\downarrow}1 \, W_{\downarrow}2 \; f\,(\xi_{\downarrow}1, (_{\downarrow}2\,) + W_{\downarrow}2^{\uparrow}2 \, f$$

DISCUSSION OF RESULTS

A brief discussion on some of the important observations made on the comparison between triangular and rectangular element discretization is presented as follows:

- Essentially, each node has two pre–eminent displacements, the vertical and horizontal displacements.
- The effective displacement at each node is actually the combination of vertical and horizontal displacement.

The effective displacements of all the nodes of both methods are tabulated in table 2, which clearly reveals the closeness between the values obtained by the methods. Evidently, the values obtained from

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triangular idealization are comparably greater than the values obtained from rectangular discretization. The concentration of the displacement is quite higher within the central region of the beam, and propagates appreciably at the axis of loading to top (at the points of loading). Conversely, the values depreciate considerably to the points of fixity.

Noteworthy, the maximum and minimum effective displacements of both methods occurred at the same place, which is at point of loading and support respectively. The maximum value from triangular idealization is 1.16489E-005, while that of rectangular idealization is 1.15063E-005. Their minimum values are zeros and occurred at node1.

DISPLACEMENT MAGNITUDE (EFFECTIVE) DISTANCE RELATIONSHIP

The graph showing the relationship between effective displacement and distance is shown below. Apparently, the graphs of the both method are closely related. Figure 4 shows the displacement magnitude–distance relationship of bottom fibre of the two methods, which reveals that the trajectory at the support is linear, explaining that the beam will be rigid at that region. On the other hand, the trajectory at spans is somewhat parabolic and climaxing at the point of loading. This representation shows that deformation of the beam will concentrate more on the span. Again, figure 5 represents the displacement magnitude–distance relationship of the top fibre which implies that, the deformation at the point of loading will be sudden which is indicated by the linearity of the trajectory of the displacement.

Table 2. Effective displacements

Node Number	Effective Displacements (m)	
	Triangular Displacement	Rectangular Displacement
Node 1	0	0
Node 2	9.99803E-07	2.49E-06
Node 3	1.92198E-06	3.77E-06
Node 4	2.80336E-06	4.39E-06
Node 5	3.67944E-06	4.8E-06
Node 6	4.582E-06	5.43E-06
Node 7	5.53534E-06	6.2E-06
Node 8	7.11836E-06	5.18E-06
Node 9	7.27842E-06	5.48E-06
Node 10	7.45951E-06	5.6E-06
Node 11	7.6629E-06	5.86E-06
Node 12	7.89663E-06	6.27E-06
Node 13	8.17568E-06	6.74E-06
Node 14	8.52622E-06	7.46E-06
Node 15	1.0191E-05	6.6E-06
Node 16	1.01697E-05	6E-06
Node 17	1.02612E-05	6.4E-06
Node 18	1.04393E-05	7.12E-06
Node 19	1.06873E-05	8.08E-06
Node 20	1.09942E-05	9.44E-06
Node 21	1.13544E-05	1.15E-05
Node 22	5.26738E-06	5.47E-06
Node 23	7.13271E-07	2.84E-06

Node 24	3.85339E-06	6.14E-06
Node 25	4.0019E-06	7.2E-06
Node 26	4.14634E-06	7.98E-06
Node 27	4.25653E-06	8.32E-06
Node 28	4.31939E-06	7.75E-06
Node 29	9.93801E-06	7.08E-06
Node 30	1.01316E-05	7.47E-06
Node 31	1.03389E-05	7.54E-06
Node 32	1.05745E-05	7.81E-06
Node 33	1.08577E-05	8.34E-06
Node 34	1.12086E-05	9.26E-06
Node 35	1.16489E-05	1.11E-05
Node 36	9.14465E-06	8.12E-06
Node 37	8.79083E-06	7.26E-06
Node 38	8.4879E-06	6.95E-06
Node 39	8.21157E-06	6.87E-06
Node 40	7.94543E-06	6.82E-06
Node 41	7.67751E-06	6.58E-06
Node 42	7.40015E-06	5.52E-06
Node 43	9.44376E-06	9.67E-06
Node 44	5.27582E-06	1.58E-06
Node 45	3.58666E-07	5.72E-06
Node 46	4.44695E-06	5.67E-06
Node 47	3.62442E-06	5.34E-06
Node 48	2.78249E-06	4.57E-06
Node 49	1.89539E-06	4.09E-06

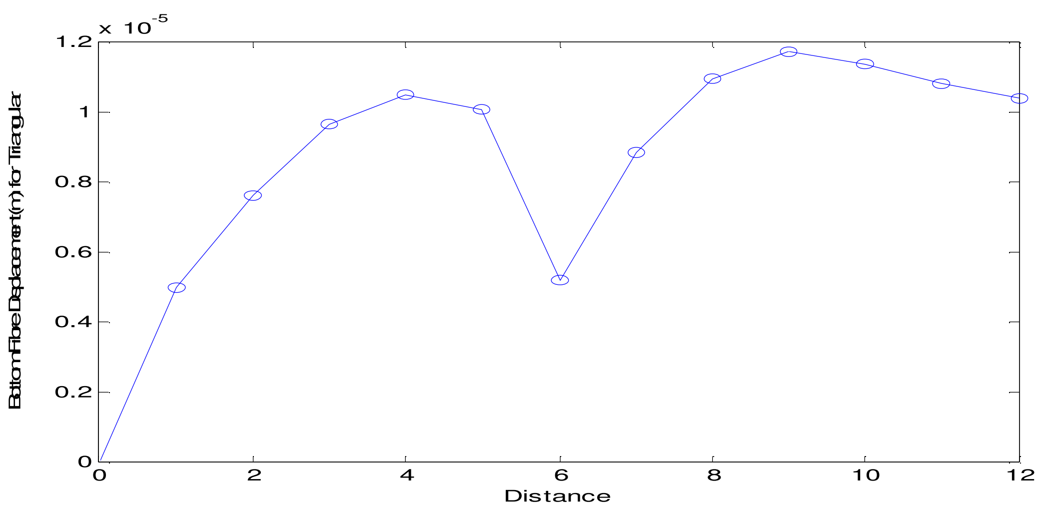

Figure 4a. Displacement magnitude (Effective)–Distance Relationship for rectangular idealization for bottom fibre.

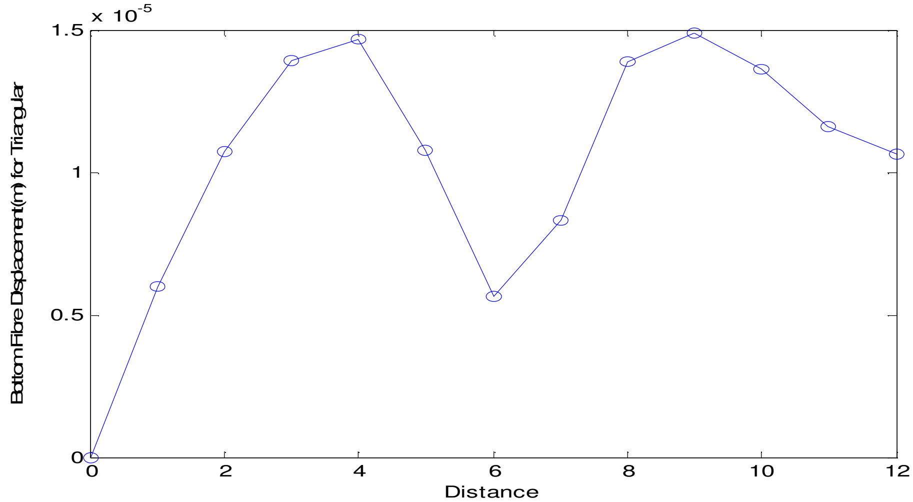

Figure 4b. Displacement magnitude (Effective) –Distance Relationship for triangular idealization for bottom fibre.

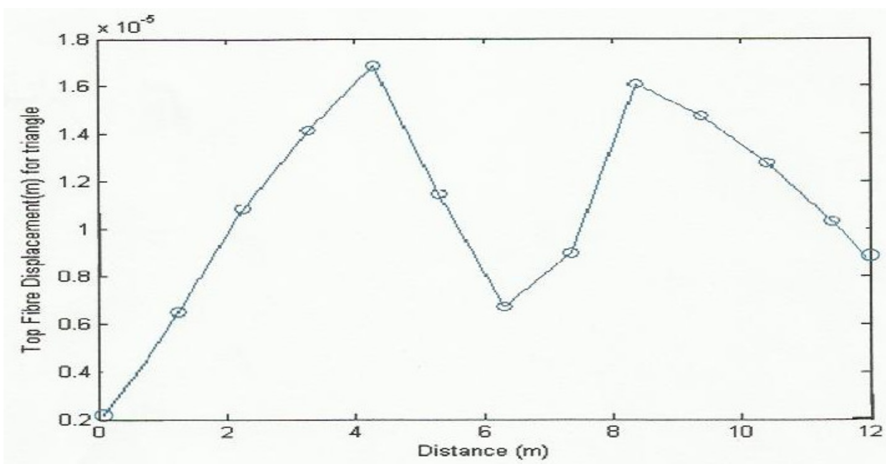

Figure 5a. Displacement magnitude (Effective) –Distance Relationship for triangular idealization for top fibre.

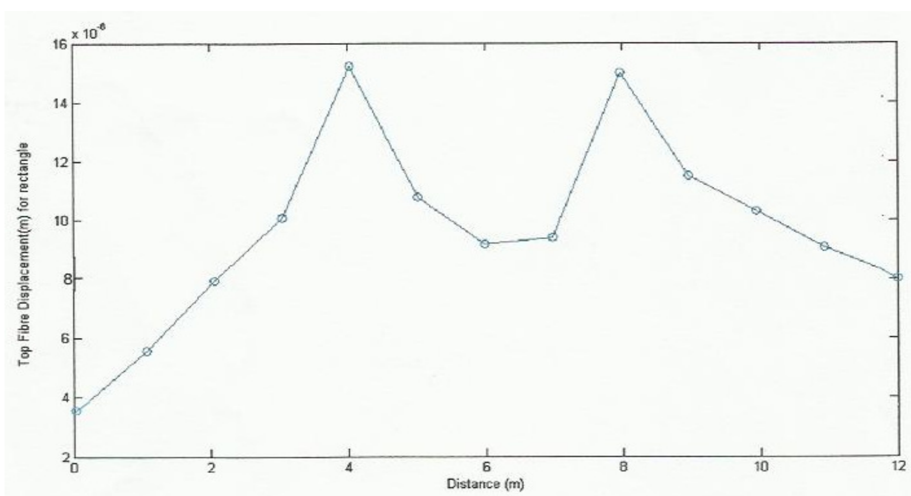

Figure 5b. Displacement magnitude (Effective) –Distance Relationship for rectangular idealization for top fibre.

CONCLUSION

In applying the finite element method (FEM) to a problem, it is traditional to discretize (subdivide) the continuum (model) into small areas of triangular or rectangular shapes (mesh). Obviously, it is clear and more straightforward to use triangular elements to model a structure purely because of the ease in the formulation of its mathematical equation. Conversely, rectangular elements involve more serious mathematical manipulation. Gauss points and weights for Gaussian Quadrature considerably make the use of triangular elements simpler and more flexible. The results obtained from rectangular elements are finer and closer to the exact solutions. It also has provision by which one can estimate the displacement, stress and other parameters at any point within the element, unlike triangular element which allows the estimation at nodal points only.

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