3D COORDINATE TRANSFORMATION USING TOTAL LEAST SQUARES

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ABSTRACT

Nowadays, there are efforts around the globe to coordinate all mapping activities using the earth-centred WGS 84. Therefore the need to transform Nigerian coordinates hitherto based on the Nigerian non-earth centred Minna Datum to the global WGS 84. This research presents a 3D coordinate transformation between the local Minna Datum and the global WGS 84 datum in Nigeria using total least squares. The Bursa-Wolf and Molodensky-Badekas similarity/conformal transformation models are used for the experiment. One hundred and ten points are selected, of which sixty points are used to compute the values of the unknown parameters while the remaining fifty points are used to compute the accuracy of the model. The experiment shows that total least squares yields better result than the least squares; and also that the Molodensky-Badekas model results are better than those of the Bursa-Wolf model.

Keywords: 3D Coordinate Transformation, Total Least Squares, Least Squares, Minna Datum, WGS 84

INTRODUCTION

The Nigerian coordinate system is based on the non-earth centred datum called "Minna Datum." The advent of the Geographic Positioning Systems (GPS) has made it very convenient to coordinate any point on the globe very accurately with a mere touch of a GPS button. In line with current practice across the globe there is needed to transform Nigerian coordinates hitherto in Minna Datum to the global earth-centred WGS 84. Because of the relatively cylindrical shape of the Nigerian geographic boundary, the Bursa-Wolf and Molodensky-Badekas similarity/conformal transformation models are chosen for this research. The Least Squares (LS) technique has been commonly used to transform coordinates from one datum to another. This research therefore introduces the Total Least Squares (TLS) approach for transforming Nigerian coordinates from Minna Datum to WGS 84.

TOTAL LEAST SQUARES

Fundamentally a LS estimate for variable $\breve{x}$ can be expressed as,

$$\breve{x} = (A^T A)^{-1} A^T b \tag{1}$$

Where $\breve{x}$ represents vector of unknown parameters; A is the design matrix; while b denotes the vector of observations or the target vector. In the case of TLS, it assumes that all the elements of the data are erroneous; this situation can be stated mathematically as,

$$b + \Delta b = (A + \Delta A)x \text{ , } rank(A) = m < n \tag{2}$$

Where, Δb is error vector of observations and ΔA is error matrix of data matrix A. Both errors are assumed independently and identically distributed with zero mean and with same variance (Dogan and Altan, 2010).

The estimation procedure is an optimisation problem given by,

$$\text{Minimize} \left\| [A;b] - [\hat{A};\hat{b}] \right\|_F \text{ . } [\hat{A};\hat{b}] \in R^{n(m+1)} \tag{3}$$

Subject to: $b + \Delta b = (A + \Delta A)\breve{x}$

Where m is the number of unknowns; and n is the number of observations. Once a minimising $[\hat{A};\hat{b}]$ is found, any $\breve{x}$ that satisfies $\hat{A}\breve{x} = \hat{b}$ is a called TLS solution and $[\Delta\hat{A};\Delta\hat{b}] = [A;b] - [\hat{A} - \hat{b}]$ is the corresponding TLS correction (Golub and Loan, 1980; Akyilmaz, 2007; Huffel and Vandewalle, 1991; Golub and Reinsch, 1970; Golub, 1973). From equation 3, $\| \ \|_F$ denotes the Frobenius norm. The basic TLS problem given in equation 3 can be solved using Singular Value Decomposition (SVD) (Golub, 1973; Golub and Loan, 1980; Huffel and Vandewalle, 1991). For the solution of $A\breve{x} \approx b$, we write the functional relation as follows:

$$[A;b]\,[\breve{x}^T;-1]^T \approx 0 \tag{4}$$

Therefore SVD of the augmented matrix $[A;b]$ is computed as follows:

$$[A;b] = U\Sigma V^T \tag{5}$$

Where, $U = [U_1;U_2]$, $U_1 = [u_1,...,u_m]$, $U_2 = [u_{m+1},...,u_n]$ and $u_i \in R^n$,
$U^T U = I_n$. $V = [v_1,...,v_m,v_{m+1}], v_i \in R^{m+1}$, V^T
$V = I_{m+1}.\Sigma = diag(\sigma_1,...,\sigma_m,\sigma_{m+1}) \in R^{n\times(m+1)}$.

In equation 5, the rank of the matrix $[A;b]$ is $m+1$ which must be reduced to m. For the purpose, Eckart-Young Minsky theorem is used (Eckart and Young, 1936). After the rank reduction, the solution of the basic TLS is obtained by,

$$[\breve{x}^T;-1]^T = \frac{-1}{V_{m+1,m+1}} v_{m+1} \tag{6}$$

If $V_{m+1,m+1} \neq 0$, then $\hat{b} = \hat{A}\breve{x} = -1/(V_{m+1,m+1})\,\hat{A}[V_{1,m+1},...,V_{m,m+1}]^T$ which belongs to the column space of $\hat{A}$, and hence $\breve{x}$ solves the basic TLS problem (Huffel and Vandewalle, 1991).

APPLICATION

This research employed the Bursa-Wolf and Molodensky-Badekas similarity transformation models. The Bursa-Wolf model is one of the most commonly used transformation methods in geodetic applications. It is a *3D conformal transformation* also known as *3D similarity transformation* or *Helmert* 3D *transformation* or *7-parameter transformation* (Andrei, 2006). Equation 7 is the mathematical representation of the Bursa-Wolf model (Andrei, 2006),

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}_{WGS\ \ 84} = \begin{bmatrix} \delta X \\ \delta Y \\ \delta Z \end{bmatrix} + \mu \cdot R(\alpha_1,\alpha_2,\alpha_3) \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{MINNA\ \ DATUM} \tag{7}$$

where, μ is the scale factor; $\alpha_1,\alpha_2,\alpha_3$ are the three rotation angles around the *x*-, *y*- and *z*-axis, respectively; $\delta X, \delta Y, \delta Z$ denote the three translations parameters;

$\cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{MINNA \quad DATUM}$ are coordinates of the first coordinate system (Minna Datum);

$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}_{WGS \quad 84}$ are coordinates of the second coordinate system (WGS 84);

R denotes the total rotation matrix which is the product of three individual rotation matrices:

$$\underset{3X3}{R} = R(\alpha_1, \alpha_2, \alpha_3) = R_3(\alpha_3) \cdot R_2(\alpha_2) \cdot R_1(\alpha_1) \tag{8}$$

$$= \begin{bmatrix} \cos\alpha_3 & \sin\alpha_3 & 0 \\ -\sin\alpha_3 & \cos\alpha_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha_2 & 0 & -\sin\alpha_2 \\ 0 & 1 & 0 \\ \sin\alpha_2 & 0 & \cos\alpha_2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_1 & \sin\alpha_1 \\ 0 & -\sin\alpha_1 & \cos\alpha_1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha_2 \cos\alpha_3 & \cos\alpha_1 \sin\alpha_3 + \sin\alpha_1 \sin\alpha_2 \cos\alpha_3 & \sin\alpha_1 \sin\alpha_3 - \cos\alpha_1 \sin\alpha_2 \cos\alpha_3 \\ -\cos\alpha_2 \sin\alpha_3 & \cos\alpha_1 \cos\alpha_3 - \sin\alpha_1 \sin\alpha_2 \sin\alpha_3 & \sin\alpha_1 \cos\alpha_3 + \cos\alpha_1 \sin\alpha_2 \sin\alpha_3 \\ \sin\alpha_2 & -\sin\alpha_1 \cos\alpha_2 & \cos\alpha_1 \cos\alpha_2 \end{bmatrix}$$

If the rotation parameters and scale factor are considered to be very small therefore equation 7 can be simplified as (Andrei, 2006),

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}_{WGS \quad 84} = \begin{bmatrix} \delta X \\ \delta Y \\ \delta Z \end{bmatrix} + \begin{bmatrix} 1+\delta\mu & \alpha_3 & -\alpha_2 \\ -\alpha_3 & 1+\delta\mu & \alpha_1 \\ \alpha_2 & -\alpha_1 & 1+\delta\mu \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{MINNA \quad DATUM} \tag{9}$$

Note that α_1, α_2, and α_3 are in radians. Equation 9 can be simplified as equations 10, 11, and 12.

$$X' - X = \delta X + X \cdot \delta\mu + Y \cdot \alpha_3 - Z \cdot \alpha_2 \tag{10}$$

$$Y' - Y = \delta Y - X \cdot \alpha_3 + Y \cdot \delta\mu + Z \cdot \alpha_1 \tag{11}$$

$$Z' - Z = \delta Z - X \cdot \alpha_2 - Y \cdot \alpha_1 + Z \cdot \delta\mu \tag{12}$$

The matrix representation for b, A, and $\breve{x}$ derived from equations 10, 11, and 12 are given in equations 13, 14, and 15 respectively (note that, b and A represent the solution for a single point),

$$b = \begin{bmatrix} X' - X \\ Y' - Y \\ Z' - Z \end{bmatrix} \tag{13}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -Z & Y & X \\ 0 & 1 & 0 & Z & 0 & -X & Y \\ 0 & 0 & 1 & -Y & X & 0 & Z \end{bmatrix} \tag{14}$$

$$\breve{x} = \begin{bmatrix} \delta X \\ \delta Y \\ \delta Z \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \delta\mu \end{bmatrix} \tag{15}$$

Equation 16 is the mathematical representation of the Molodensky-Badekas model (Andrei, 2006),

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}_{WGS\ 84} = \begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix}_{MINNA\ DATUM} + \begin{bmatrix} \delta X \\ \delta Y \\ \delta Z \end{bmatrix} + \mu \cdot R(\alpha_1, \alpha_2, \alpha_3) \cdot \begin{bmatrix} X - \bar{X} \\ Y - \bar{Y} \\ Z - \bar{Z} \end{bmatrix}_{MINNA\ DATUM} \tag{16}$$

$\bar{X}$, $\bar{Y}$, $\bar{Z}$ are the centroid of X, Y, Z coordinates for points in the local Minna Datum; where

$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$; $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$; and $\bar{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i$.

If the rotation parameters and scale factor are considered to be very small, equation 16 becomes (Andrei, 2006),

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}_{WGS\ 84} = \begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix} + \begin{bmatrix} \delta X \\ \delta Y \\ \delta Z \end{bmatrix} + \begin{bmatrix} 1+\delta\mu & \alpha_3 & -\alpha_2 \\ -\alpha_3 & 1+\delta\mu & \alpha_1 \\ \alpha_2 & -\alpha_1 & 1+\delta\mu \end{bmatrix} \cdot \begin{bmatrix} X - \bar{X} \\ Y - \bar{Y} \\ Z - \bar{Z} \end{bmatrix}_{MINNA\ DATUM} \tag{17}$$

Equation 17 can be simplified as equations 18, 19, and 20,

$$X' - X - (X - \bar{X}) = \delta X + (X - \bar{X}) \cdot \delta\mu + (Y - \bar{Y}) \cdot \alpha_3 - (Z - \bar{Z}) \cdot \alpha_2 \tag{18}$$

$$Y' - Y - (Y - \bar{Y}) = \delta Y - (X - \bar{X}) \cdot \alpha_3 + (Y - \bar{Y}) \cdot \delta\mu + (Z - \bar{Z}) \cdot \alpha_1 \tag{19}$$

$$Z' - Z - (Z - \bar{Z}) = \delta Z - (X - \bar{X}) \cdot \alpha_2 - (Y - \bar{Y}) \cdot \alpha_1 + (Z - \bar{Z}) \cdot \delta\mu \tag{20}$$

The matrix representation for b and A derived from equations 18, 19, and 20 are given in equations 21 and 22 respectively (note that, b and A represent the solution for a single point),

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -(Z-\bar{Z}) & Y-\bar{Y} & X-\bar{X} \\ 0 & 1 & 0 & Z-\bar{Z} & 0 & -(X-\bar{X}) & Y-\bar{Y} \\ 0 & 0 & 1 & -(Y-\bar{Y}) & X-\bar{X} & 0 & Z-\bar{Z} \end{bmatrix} \tag{21}$$

$$b = \begin{bmatrix} X' - X - (X - \bar{X}) \\ Y' - Y - (Y - \bar{Y}) \\ Z' - Z - (Z - \bar{Z}) \end{bmatrix} \tag{22}$$

The solution of $\breve{x}$ for the Molodensky-Badekas model is same as equation 15 of the Bursa-Wolf model. One hundred and ten points whose geographic coordinates are known in both WGS 84 and

Minna Datum were chosen all over Nigeria (Figure 1). Sixty points were used as training data while fifty points served as test data (Figure 2).

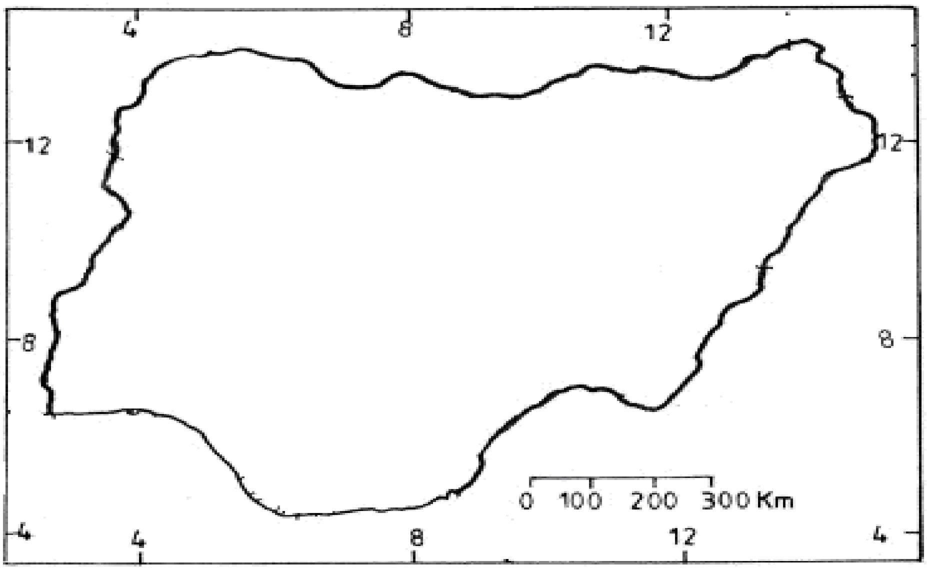

Figure 1: Map of Nigeria (y-axis=latitude; x-axis=longitude)

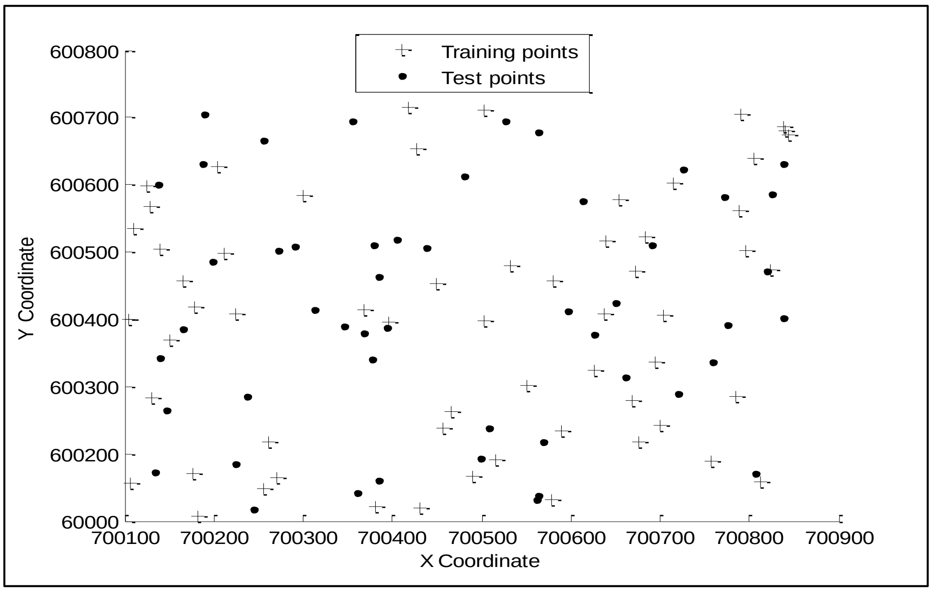

Figure 2: Planimetric representation of training and test points

Because the chosen points are in geographic coordinate system they were converted to geocentric coordinates *X*, *Y*, *Z* using the formulas,

$$X = (v + h)\cos\phi\cos\lambda \tag{23}$$

$$Y = (v + h)\cos\phi\sin\lambda \tag{24}$$

$$Z = ((1 - e^2)v + h)\sin\phi \tag{25}$$

Where $v = a/(1 - e^2\sin^2\phi)^{1/2}$ and $e^2 = 2f - f^2$. For WGS 84, a = 6378137.0000 and f = 1/298.257223563; for Minna Datum, a = 6378249.1450 and f = 1/293.46500000.

The TLS modeling was implemented in MATLAB. Sixty points were used to compute the unknown parameters δX, δY, δZ, α_1, α_2, α_3, and $\delta\mu$; while fifty points were used to compute the error or

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discrepancy between the actual coordinates and the computed coordinates. The TLS solution for δX, δY, δZ, α_1, α_2, α_3, and $\delta\mu$ is given in Table 1.

Table 1: TLS solution for $\breve{x}$ using the Bursa-Wolf and Molodensky-Badekas models

	Bursa-Wolf model	Molodensky-Badekas model
δX	-726.4817	-727.7382
δY	-498.4422	-487.1422
δZ	-528.3525	-527.9492
α_1	0.1074	0.1050
α_2	-0.1606	-0.1615
α_3	0.2331	0.2507
$\delta\mu$	1.1087	1.1085

TLS result was compared with the LS method (see Tables 1 and 2). TLS and LS results were compared based on the Root Mean Square Deviation (RMSD) estimates. The RMSD for X, Y, and Z can be computed as,

$\sqrt{\frac{\sum_{i=1}^{n}(X_i^A - X_i^C)}{n}}$, $\sqrt{\frac{\sum_{i=1}^{n}(Y_i^A - Y_i^C)}{n}}$, and $\sqrt{\frac{\sum_{i=1}^{n}(Z_i^A - Z_i^C)}{n}}$ respectively. Where n =50; X_i^A, Y_i^A, Z_i^A are the actual geocentric coordinates, while X_i^C, Y_i^C, Z_i^C are the computed geocentric coordinates.

An inverse solutions for ϕ, λ, and h were obtained using equations 26-29.

$$\phi = a\tan(Z + e^2 v \sin\phi)/(X^2 + Y^2)^{1/2} \tag{26}$$

Therefore for the i th solution of ϕ, equation 29 becomes,

$$\phi_i = a\tan(Z + e^2 v \sin\phi_{i-1})/(X^2 + Y^2)^{1/2} \tag{27}$$

$$\lambda = a\tan(Y/X) \tag{28}$$

$$h = X \sec\lambda \sec\phi - v \tag{29}$$

The results using Bursa-Wolf and Molodensky-Badekas models are presented in Tables 2 and 3.

Table 2: Computed RMSD for Bursa-Wolf model

	RMSD		Maximum positive error		Maximum negative error	
	TLS	LS	TLS	LS	TLS	LS
X	0.0442m	0.0536m	0.0652m	0.0767m	-0.0711m	-0.0920m
Y	0.0498m	0.0573m	0.0687m	0.0799m	-0.0775m	-0.0968m
Z	0.0502m	0.0638m	0.0704m	0.0821m	-0.0788m	-0.0896m
ϕ	$0.0252''$	$0.0305''$	$0.0557''$	$0.0630''$	$-0.0597''$	$-0.0611''$
λ	$0.0307''$	$0.0398''$	$0.0572''$	$0.0666''$	$-0.0603''$	$-0.0754''$
h	0.0515m	0.0643m	0.0786m	0.0818m	-0.0773m	-0.0855m

Table 3: Computed RMSD for Molodensky-Badekas model

	RMSD		Maximum positive error		Maximum negative error	
	TLS	LS	TLS	LS	TLS	LS
X	0.0398m	0.0486m	0.0556m	0.0687m	-0.0699m	-0.0800m
Y	0.0379m	0.0414m	0.0552m	0.0632m	-0.0611m	-0.0871m
Z	0.0444m	0.0521m	0.0680m	0.0764m	-0.0642m	-0.0788m
ϕ	$0.0183''$	$0.0231''$	$0.0429''$	$0.0585''$	$-0.0420''$	$-0.0542''$
λ	$0.0249''$	$0.0278''$	$0.0483''$	$0.0526''$	$-0.0537''$	$-0.0654''$
h	$0.0489\,\mathrm{m}$	$0.0565\,\mathrm{m}$	$0.0638\,\mathrm{m}$	$0.0784\,\mathrm{m}$	$-0.0628\,\mathrm{m}$	$-0.0741\,\mathrm{m}$

CONCLUSION

The Molodensky-Badekas model results were better than those of the Bursa-Wolf model. The adjusted parameters of the Bursa-Wolf model are highly correlated when the network of points used to determine the parameters covers only a small portion of the earth. The Molodensky-Badekas model (Badekas, 1969) removes the high correlation between parameters by relating the parameters to the centroid of the network (Andrei, 2006). TLS yielded equivalent results as LS; nonetheless the TLS results were better than those of LS. The result of this work showed that the TLS is a viable alternative to the LS.

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