SOME NEW DERIVATIVE FREE METHODS FOR SOLVING NONLINEAR EQUATIONS

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ABSTRACT

This paper proposes two new iterative methods for solving nonlinear equations. In comparison to the classical Newton's method, the new proposed methods do not use derivatives; furthermore only two evaluations of the function are needed per iteration. Using the methods proposed, when the starting value is selected close to the root, the order of convergence is 2. The development of the method allows you to achieve classical methods such as secant and Steffensen's as an alternative to the usual process. The numerical examples show that the proposed methods have the same performance as Newton's method with the advantage of being derivative free. In comparison to other methods which are derivative free, these methods are more efficient.

Keywords: Iterative method, nonlinear equations, derivative free method, Newton´s method.

INTRODUCTION

Solving equations is one of the most important problems in numerical analysis. The importance of this kind of problem solving has contributed to the development of several iterative methods for solving nonlinear equations. Newton's famous method for finding the root of a nonlinear equation uses the iterative method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

starting from an initial value x_0. Newton's method is an important basic method with order of convergence 2.

Many authors have studied and proposed multiple methods for solving nonlinear equations with high order convergence (Chun and Ham, 2007), (Kou, 2007) and (Heydari et al., 2011). However, they have to use high order derivatives which is a serious disadvantage. Sometimes the applications of the iterative methods which depend on derivatives are restricted in engineering and sciences.

There are numerous papers about iterative methods without the use of derivatives at the expense of additional evaluation of the function (Weerakoon and Fernando, 2000), (Jain, 2007), (Cordero et al., 2010) and (Nusrat and Moin, 2012). To overcome this we present and analyze two methods for solving nonlinear equations that do not require the derivative of the function and use only two evaluations of the function per iteration.

Our motivation to pursue this new technique is Muller´s method, which uses 3 values to obtain a parabola that approximates the original function. In this paper, we will use a particular quadratic polynomial. Using Newton's method we can obtain an approximation of the root of a polynomial. This root will be the new approximation to the original root.

As a result we obtain an iterative formula that for each different case provides a different iterative method, in particular, both iterative methods such as the Secant's and Steffensen´s.

DEVELOP OF THE METHODS

Consider the nonlinear function $f(x)$, let r be a zero of $f(x)$ and x_0 a starting value close to the root r. Let $-1<\alpha,\beta<1$ be sufficiently small and define

$$K=f(x_0+\alpha), L=f(x_0), M=f(x_0+\beta). \tag{1}$$

Take the particular two interpolating polynomials

$$q(x)=A(x-x_0)+B, \tag{2}$$

$$p(x)=a(x-x_0)(x-x_0-\beta)+b(x-x_0)+c, \tag{3}$$

and consider the following system of two equations:

$$K=q(x_0+\alpha)=A\alpha+B, L=q(x_0)=B, \tag{4}$$

$$K=p(x_0+\alpha)=a\alpha(\alpha-\beta)+b\alpha+c, L=p(x_0)=c, M=p(x_0+\beta)=b\beta+c. \tag{5}$$

Solving both systems by simple substitutions we have

$$B=L, A=\frac{K-L}{\alpha}, \tag{6}$$

$$c=L, b=\frac{M-L}{\beta}, a=\frac{\beta(K-L)-\alpha(M-L)}{\alpha\beta(\alpha-\beta)}. \tag{7}$$

Solving (2) for $q(x)=0$, substituting (6) and using Newton's method for $p(x)$ in (3) with x_0 as an approximation and substituting (7) we have

$$r\approx x_0-\frac{B}{A}=x_0-\frac{\alpha L}{K-L}, \tag{8}$$

$$r\approx x_0-\frac{p(x_0)}{p'(x_0)}=x_0-\frac{L}{b-a\beta}=x_0-\frac{\alpha\beta(\alpha-\beta)L}{\alpha^2(M-L)-\beta^2(K-L)}. \tag{9}$$

Substituting (1) in the above results we get the following methods

$$x_{n+1}=x_n-\frac{\alpha f(x_n)}{f(x_n+\alpha)-f(x_n)}, \tag{10}$$

$$x_{n+1}=x_n-\frac{\alpha\beta(\alpha-\beta)f(x_n)}{\alpha^2\left(f(x_n+\beta)-f(x_n)\right)-\beta^2\left(f(x_n+\alpha)-f(x_n)\right)}, \tag{11}$$

for suitable choices for α,β.

The choice of α,β is the key to the methods. In this way various methods for each choice of α,β can be obtained. The development of the method shows an easy way to obtain different known methods and also a simple way to gather results about the order of convergence.

For (10)

i) If $\alpha=x_{n-1}-x_n$, then $x_{n+1}=x_n-\frac{(x_n-x_{n-1})f(x_n)}{f(x_n)-f(x_{n-1})}$, we get secant method which has an order of convergence 1.628. (12)

ii) If $\alpha=f(x_n)$, then $x_{n+1}=x_n-\frac{f^2(x_n)}{f(x_n+f(x_n))-f(x_n)}$, we get the Steffensen's method which has an order of convergence 2. (13)

For (11)

iii) If $\beta=-\alpha=f(x_n)$, then $x_{n+1}=x_n-\frac{2f^2(x_n)}{f(x_n+f(x_n))-f(x_n-f(x_n))}$, we get the (14) method that uses the central difference approximation for $f'(x_n)$ and which has an order of convergence 2.

We propose the following derivative free method (N1) which has an order of convergence 2 for $\alpha = -\beta = x_n - x_{n-1}$:

$$x_{n+1} = x_n - \frac{2(x_n - x_{n-1}) f(x_n)}{f(2x_n - x_{n-1}) - f(x_{n-1})}. \tag{15}$$

The derivative free method (N2) has an order of convergence 2 by substituting $\alpha = x_{n-1} - x_n, \beta = f(x_n)$:

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1}) f^2(x_n)(x_n - x_{n-1} + f(x_n))}{(x_n - x_{n-1})^2 f(x_n + f(x_n)) + f^2(x_{n-1})(f(x_n) - f(x_{n-1}))}. \tag{16}$$

We consider the definition of the efficiency index (Traub, 1977) as $I = p^{1/w}$, where p is the order of the method and w is the number of function evaluations per iteration required by the method. It can been seen that methods (15) and (16) requires two function evaluations per iteration with an efficiency index $\sqrt{2} \approx 1.414$. The next section describes in detail the convergence analysis of the proposed methods.

CONVERGENCE ANALYSIS

In this section we will present the convergence analysis by giving mathematical proof for the order of convergence of the methods defined by (15) and (16).

Consider the nonlinear function $f(x)$ and let r be a zero of $f(x)$. Given x_n in the methods, consider the polynomial

$$p(x) = a(x - x_n)(x - x_n - \beta) + b(x - x_n) + c = a(x - x_n)^2 + (b - a\beta)(x - x_n) + c.$$

By using generalized Rolle's Theorem and (7), we know that

$$a = \frac{\beta(K - L) - \alpha(M - L)}{\alpha\beta(\alpha - \beta)} = \frac{f''(\xi_3)}{2},$$

with ξ_3 between $x_n, x_n + \alpha, x_n + \beta$.

Also, using the error for an interpolating polynomial $p(x)$ we have

$$f(x) = \frac{f''(\xi_3)}{2}(x - x_n)^2 + (b - a\beta)(x - x_n) + c + \frac{f'''(\xi_2)}{6}(x - x_n)(x - x_n - \alpha)(x - x_n - \beta), \tag{17}$$

with ξ_2 between $x_n, x_n + \alpha, x_n + \beta$.

Theorem 1

Let α be a single root of a sufficiently differentiable function $f: I \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to α, then the iterative method (15) converge on the root α, the order of convergence is 2 and the asymptotic constant satisfy $2f'(r)k^2 - kf''(r) - f'''(r) = 0$.

Proof The assumption is that we have chosen x_{n-1}, x_n in the method. We assume that the points x_n, x_{n-1} lie in a neighborhood of a root r. Using (17) and $\alpha = -\beta = x_n - x_{n-1}$ we obtain

$$f(x) = \frac{f''(\xi_3)}{2}(x - x_n)^2 + (b - a\beta)(x - x_n) + c + \frac{f'''(\xi_2)}{2}(x - x_n)(x - 2x_n + x_{n-1})(x - x_{n-1}).$$

Therefore, for $x = x_{n+1}$ and with the Taylor's expansion of $f(x_{n+1})$ about r we have

$$f'(\xi_1)\varepsilon_{n+1}=\frac{f''(\xi_3)}{2}(\varepsilon_{n+1}-\varepsilon_n)^2+\frac{f'''(\xi_2)}{2}(\varepsilon_{n+1}-\varepsilon_n)(\varepsilon_{n+1}-2\varepsilon_n+\varepsilon_{n-1})(\varepsilon_{n+1}-\varepsilon_{n-1}),$$

where $\varepsilon_{n+1}=x_{n+1}-r,\varepsilon_n=x_n-r$.

Suppose that ε_{n+1} is asymptotic to $k\varepsilon_n, p>1$. Consequently, by expressing the above equation in terms of ε_{n-1}, we determine that

$$k^{p+1}\varepsilon_{n-1}^{p^2}f'(\xi_1)$$

is asymptotic to

$$\frac{f''(\xi_3)}{2}k^2\varepsilon_{n-1}^{2p}\left(k^p\varepsilon_{n-1}^{p^2-p}-1\right)^2+\frac{f'''(\xi_2)}{2}k\varepsilon_{n-1}^{p+2}\left(k^p\varepsilon_{n-1}^{p^2-p}-1\right)\left(k^{p+1}\varepsilon_{n-1}^{p^2-1}-2k\varepsilon_{n-1}^{p-1}+1\right)\left(k^{p+1}\varepsilon_{n-1}^{p^2-1}-1\right).$$

In order to satisfy the previous asymptotic equation there are two possibilities. As the equations below show, in both cases, it is evident that p must be a positive root of

$$p^2-p-2=0, \text{ or}$$

$$p^2-2p=0.$$

In both cases, the positive root is $p=2$. And finally, in the limit when $n\to\infty$, k must satisfy

$$2f'(r)k^2-kf''(r)-f'''(r)=0.$$

Therefore,

$$\lim_{n\to\infty}\frac{\varepsilon_{n+1}}{\varepsilon_n^2}=k,$$

which shows that the new method (15) has a quadratic convergence.

Theorem 2

Let α be a single root of a sufficiently differentiable function $f:I\to\mathbb{R}$ for an open interval I. If x_0 is sufficiently close to α, then the iterative method (16) converge on the root α, the order of convergence is 2 and the asymptotic constant is $\frac{f''(r)}{2f'(r)}$.

Proof: Similar to the above theorem, we assume that we have chosen x_{n-1},x_n in the method. We surmise that the points x_n,x_{n-1} lie in a neighborhood of a root r. Using (17) and $\alpha=x_{n-1}-x_n,\beta=f(x_n)$ we obtain

$$f(x)=\frac{f''(\xi_3)}{2}(x-x_n)^2+(b-a\beta)(x-x_n)+c+\frac{f'''(\xi_2)}{2}(x-x_n)(x-x_{n-1})(x-x_n-f(x_n)).$$

Therefore, for $x=x_{n+1}$ and with the Taylor's expansion of $f(x_{n+1})$ and $f(x_n)$ about r we have

$$f'(\xi_1)\varepsilon_{n+1}=\frac{f''(\xi_3)}{2}(\varepsilon_{n+1}-\varepsilon_n)^2+\frac{f'''(\xi_2)}{2}(\varepsilon_{n+1}-\varepsilon_n)(\varepsilon_{n+1}-\varepsilon_{n-1})(\varepsilon_{n+1}-\varepsilon_n+\varepsilon_nf'(\xi_3)),$$

where $\varepsilon_{n+1}=x_{n+1}-r,\varepsilon_n=x_n-r$.

Suppose that ε_{n+1} is asymptotic to $k\varepsilon_n, p>1$. Consequently, by expressing the above equation in terms of ε_{n-1}, we obtain that

$$k^{p+1}\varepsilon_{n-1}^{p^2}f'(\xi_1)$$

is asymptotic to

$$k^2\varepsilon_{n-1}^{2p}\left[\frac{f''(\xi_3)}{2}\left(k^p\varepsilon_{n-1}^{p^2-p}-1\right)^2+\frac{f'''(\xi_2)}{2}\varepsilon_{n-1}\left(k^p\varepsilon_{n-1}^{p^2-p}-1\right)\left(k^{p+1}\varepsilon_{n-1}^{p^2-1}-1\right)\left(k^p\varepsilon_{n-1}^{p^2-p}-1+f'(\xi_3)\right)\right].$$

In order to satisfy the previous asymptotic equation, it is evident that p has to be a positive root of

$$p^2-2p=0,$$

that is, $p=2$ and finally in the limit when $n \to \infty$, $k = \dfrac{f''(r)}{2f'(r)}$.

Therefore,

$$\lim_{n\to\infty} \frac{\varepsilon_{n+1}}{\varepsilon_n^2} = \frac{f''(r)}{2f'(r)}.$$

Which shows that the new method (16) has an order of convergence 2.

NUMERICAL EXAMPLES

In this section we present some numerical examples by employing (N1) and (N2) methods and compare them with Newton's method (NM), Steffensen's method (SM), Nusrat Moin methods (NM1), (NM2) and Cordero's method (CM). All tests are performed using double arithmetic precision on MathLab. Table 1 shows the number of iterations and the number of functional evaluations. The stop criterion is $|x_n - x_{n-1}| < 10^{-16}$. In (N1) and (N2) we consider the initial approximation as x_1 and take $x_0 = x_1 + 10^{-5}$ to obtain x_2. The choice of 10^{-5} only depends on the order of convergence.

The functions used in Table 1 are common and appear in (Weerakoon and Fernando, 2000), (Cordero et al., 2010) and (Nusrat and Moin, 2012).

$f_1(x) = (x-1)^3 - 2,$ $\quad r = 2.2599210498948734$

$f_2(x) = x^3 + 4x^2 - 10,$ $\quad r = 1.3652300134140969$

$f_3(x) = \sin^2(x) - x^2 + 1,$ $\quad r = 1.4044916482153411$

$f_4(x) = \sin(x) - \dfrac{x}{2},$ $\quad r = 1.8954942670339809$

$f_5(x) = e^x - 3x^2,$ $\quad r = 0.9100075724887091$

Table 1. Comparison of the methods.

$f(x)$	x_0	Iterations							NOFE						
		NM	SM	NM1	NM2	CM	N1	N2	NM	SM	NM1	NM2	CM	N1	N2
f_1	1.85	6	6	4	6	4	6	5	12	12	16	12	16	18	10
f_2	2	5	5	3	4	3	5	4	10	10	12	12	12	10	8
f_3	1.5	4	4	2	3	2	4	4	8	8	8	9	8	8	8
f_4	2	4	4	2	3	2	4	4	8	8	8	9	8	8	8
f_5	0.5	6	6	3	4	3	5	5	12	12	12	12	12	10	10

REMARKS

The methods obtained in (Jain, 2007), (Weerakoon and Fernando, 2000), (Cordero et al., 2010) and (Nusrat and Moin, 2012) are combinations of results from methods (12), (13) and (14). The order of convergence increases at the same rate as the number of functional evaluations. In this way, the efficiency index does not change. Looking at methods that use derivatives of the function, (Chun and Ham, 2007), (Kou, 2007) , (Heydari et al., 2011), the main supposition is that all evaluations of the function, including the derivatives, have the same computational cost. In this case, a greater efficiency index is obtained. In general, we know that the derivative has a higher computational cost. In this sense, Newton's method is the main reference for all types of methods that can be described. For this reason, the fact that the new methods obtain the same efficiency index as Newton's without the use of derivatives is an important advantage.

CONCLUSIONS

In this paper, we have proposed two iterative methods with an order of convergence 2. The methods use only two evaluations of the function and avoid the use of derivatives of the function. The efficiency index is 1.414 for both methods, which is equal to the Newton's and Steffensen's efficiency index. The numerical examples show that these methods have equal or better performance than the classical methods mentioned above and even some new methods of high order of convergence. The process provides a means to create alternative derivative free methods which have a different order of convergence.

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