DETERMINISM VERSUS PREDICTABILITY IN THE CONTEXT OF POINCARE'S WORK ON THE RESTRICTED 3-BODY PROBLEM[1]

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ABSTRACT

It is exposed the Poincare's work (1880's) in the system defining the restricted 3-body problem which led to the discovery of a special kind of behavior – the dynamical instability. Contrary to the widespread belief that the Deterministic Chaos Theory began with the computational work of Lorenz (1960's), the Poincare's theoretical research was sufficiently clear about the existence of chaotic deterministic behavior. Until the time of Poincare, there was a tacit assumption that the uncertainty in the output does not arise from any randomness in the dynamical laws, since they are completely deterministic, but rather from the lack of the infinite accuracy in the initial conditions. In this paper it is emphasized that the issues of determinism and predictability are distinct.

Keywords: *dynamical instability, Chaos Theory, Determinism, unpredictability*

INTRODUCTION

We live in a world in which insignificant details can have a great impact. Very tiny changes in the initial conditions of an event can have substantial (sometimes even dramatic) effects on subsequent results.

However, for centuries the prevailing view of the Universe was that it "runs like clockwork", and its running can be numerically predicted from a given set of initial conditions. As it is shown, this viewpoint is naive: it was verified in many real-world phenomena that small differences in the initial conditions of a process can have a significant effect on the final outcome.

The first mathematical tools necessary to understand this kind of real-world phenomena were given by Poincare's work in the restricted *3*-body problem and come from the field of chaotic dynamics in mathematics. This work is a response to a challenge posedin1885by King Oscar II of Sweden.

However, although the Poincare's theoretical research was sufficiently clear from the existence of chaotic deterministic behavior, the evidence to the scientific community provided by his work was only possible due to the use of a computer through the work of the mathematician and meteorologist Edward Lorenz almost 75years later.

In order to understand the importance of Poincare's work in dynamical systems and the historical guidelines of the development of Deterministic Chaos Theory, it is essential to take into account the new tools that technology provided to the progress of Science. In turn, the concept of Deterministic

[1] The initial findings of paper were presented at the IJAS Conference in Rome (Oct. 31 – 03 Nov., 2011).

Chaos Theory has extensive applications in current research in natural sciences, engineering, financial markets and information systems.

THE RESTRICTED 3-BODY PROBLEM

Difficulty on finding the solutions of *n*-body problem with $n \geq 3$

The *n*-body problem consists on determining the possible motions of *n* bodies moving under no influence other than that of their mutual gravitation:

> *"Given an initial set of data with the initial positions $s_i(0)$, masses m_i and velocities $s_i'(0)$ of n bodies ($i=1,2,\ldots,n$), with $s_i(0) \neq s_j(0)$ for all mutually distinct i and j, to determine the motions of the n bodies, and to find their positions at other times t, in accordance with the laws of classical mechanics."*

The first complete mathematical formulation of the *n*-body problem appeared at the Book 1 of *Principia* (1687) sinceit was very important to understand the motion of celestial bodies in accordance with the Newton's laws of classical mechanics. Newton proved that celestial bodies can be modeled as mass points. Expressing the gravitational interactions which determine the motion of celestial bodies, and using his second law of motion, he obtained an initial-value problem of autonomous ordinary differential equations (ODE).

The system of *n* bodies is an example of deterministic dynamical system (DDS). It is a continuous DDS. It contains $6\times n$ variables, since each mass point is represented by *3* (Euclidean) space-components and *3* velocity-components. Bruns (1887) proved that the ten classical integrals/solutions (*3* for the positions of center of mass, *3* for the velocity of center of mass, *3* for the angular momentum and *1* for the energy) are the only algebraically linearly independent of this system with $6\times n$ degrees of freedom. This only allows the reduction to $6\times n-10$ variables. For $n=2$, the number of variables is reduced to *2*. In this case the DDS (of *2* ODE) is integrable. It was completely solved by Johann Bernoulli in early 18th century, each body travels along a conic section which has a focus at the centre of mass of the system.

Historically, the case $n=3$ is the most interesting and studied, mainly because the *3*-body problem models the Earth-Moon-Sun system. Bruns's work reduces the number of variables of the DDS from *18* to *8*. Many of the early attempts to solve the *3*-body problem were quantitative in order to find (exact or approximate) explicit solutions for particular situations. D'Alembert (1747) and Clairaut (1747) developed a long-standing rivalry, both attempted to analyze the problem in some degree of generality. Euler (1767), Lagrange (1772), Jacobi (1836), Delaunay (1860) and Hill (1878), among others, also had study the *3*-body problem. But the *n*-body problem is surprisingly difficult to solve for $n\geq 3$, even in the so called restricted *3*-body problem.

MATHEMATICS COMPETITION SPONSORED BY OSCAR II

The problem of describing the interrelated motion of more than *2* bodies remained so difficult in the late 19th century that the King Oscar II of Sweden, advised by the mathematician Mittag-Leffler, established an award for its solution. The challenge was:

> "*Given a system of arbitrarily many mass points which attract each other according to Newton's law, under the assumption that no 2 points ever collide, try to find a representation of the coordinates of each point as a series in a variable which is some known function of time and, for all of whose values, the series converges uniformly."*

Poincare was one of the scientists who worked in this challenge.

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POINCARE'S CONTRIBUTION AND EMERGENCE OF DETERMINISTIC CHAOS THEORY

Poincare made a research on the DDS defining the restricted *3*-body problem where the mass of one of the bodies is infinitely small. A particular case of this restricted problem was analytically solved by Lagrange assuming that *2* bodies are in circular orbits and the third has a negligible mass.

Poincare used his doctoral dissertation on mathematics ("Sur les propriétés des fonctions définies par les équations différences", 1879, advisor Charles Hermite) in which he devised a new way of studying the general geometric properties of functions defined by ODE, based on the topology and using the Lobachevsky's non-Euclidean geometry. He clearly saw that his work could be used to model the behavior of multiple bodies in free motion within the solar system, where the essential question is about the stability and qualitative properties of planetary orbits and not about the numerical solution. It was with this conviction that Poincare devoted himself to the resolution of the n-body problem, for $n \geq 3$, defined by a second order system of n ODE.

In 1887 Poincare proved that the problem did not have a solution. Instead he found the dynamical instability, currently designed by chaotic behavior. It is a special kind of behavior characterized by the existence of orbits in the system which are non periodic, and yet not forever increasing to infinity nor approaching a equilibrium point. The evolution of such a system is often chaotic in the sense that

> *"It may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible.",*

As written by Poincare in *Science and Method* (1903). Poincare's work was a proof that for some systems, like the DDS which models the restricted 3-body problem, the only way to obtain predictions with significant degree of accuracy is to determine the initial conditions with absolutely infinite precision, because one of the properties of these systems is sensitivity to initial conditions. One of the judges of the competition, the distinguished mathematician Karl Weierstrass said*:*

> *"This work cannot indeed be considered as furnishing the complete solution of the question proposed, but it is nevertheless of such importance that its publication will inaugurate a new era in the history of celestial mechanics."*

Weierstrass referred to the original important ideas which led to the Deterministic Chaos Theory.

Poincare published his fundamental work in "Les Méthodes Nouvelles de la Mécanique Céleste" (1892-1899) and "Leçons de Mécanique Céleste" (1905-1910).

DETERMINISM AND (UN)PREDICTABILITY

Legacy of Newtonian physics which inspired Determinism

A common belief toward the end of the 19th century was that mathematics (when properly combined with logic) can be used to obtain an exact description of the world around us. So, if we can know the state of the universe at one moment, and write out all the laws that govern it, we can accurately predict its state at any other moment. This doctrine – the Determinism - states that the cause-and-effect rules completely govern all the motion and structure on the material level. So, in principle, every event can be completely predicted in advance, or in retrospect.

Determinism was inspired by Newtonian physics composed by a concise set of principles. Newton's laws, being expressed as mathematical equations and not simply as ordinary sentences, are completely deterministic. ODE of a given DDS connect the measurements at a given initial moment to the values

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at a later or earlier moment and allow to determine exactly what to expect, given a set of initial conditions. Two nearly-indistinguishable sets of initial conditions will always produce identically the same behavior at any moment in the future or past.

Role of measurement accuracy of initial conditions: the "shrink-shrink" rule

Until the time of Poincare, there was an implicit assumption among almost all scientists:

> *"If you could decrease the uncertainty in the real measurements then any imprecision in the prediction would decrease in the same way."*

This idea is called the "shrink-shrink" rule. Uncertainty in the final predictions was a minor problem because it was assumed that it does not arise from any properties of the dynamical laws (since they are completely deterministic). It was theoretically possible to obtain nearly-perfect predictions for the behavior of any DDS. By putting more and more precise inputs, we got more precise outputs for any later or earlier moment.

"Shrink-shrink " rule is not valid in the 3-body DDS

Although certain DDS did indeed obey the "shrink-shrink " rule, the 3-body system did not. Any imprecision at all in the initial conditions, no matter how small, result after a short period of time in an uncertain deterministic prediction. If a chaotic output is generated by one set of initial conditions and then they are changed, even just a little bit (like a perturbation), the output will change over time. This expansion (blowing up) of tiny uncertainties in the initial conditions into enormous uncertainties in the outcomes remained even if the initial uncertainties were decreased to smallest imaginable size.

So, there was a conflict with the modern science. That is, although the system follows deterministic rules, its time evolution appears random. The system is predictable in principle but unpredictable in practice. Even using perfect measuring devices, it is impossible to record a real measurement with infinite precision, since this requires to display an infinite number of digits. In dynamics, the presence of uncertainty in any real measurement means that the initial conditions in studying any DDS cannot be specified to infinite accuracy.

DETERMINISTIC NATURE OF THE CHAOTIC DYNAMICAL SYSTEMS

In common language, chaos means disorder or randomness. In mathematics there is much more: it is a very specific kind of unpredictability, that is a deterministic behavior where the "shrink-shrink" rule is not valid.

It is also common to think that chaotic DDS are random. However, they are deterministic systems governed by mathematical equations, and so they are completely predictable given perfect knowledge of the initial conditions. There is no inherent randomness in this regard. Such accurate knowledge is never attainable in real life because slight errors are intrinsic to any real measurement.

Discovery of Chaos had important consequences for modeling real-world systems. Given a DS exhibiting random behavior, a useful question is whether the DS is better approximated using a deterministic model that allows chaotic dynamics or, alternatively, by a stochastic model, or yet by a mixed deterministic/stochastic model.

Chaos is the rule and not the exception. There are chaotic examples in economics, fluid dynamics, optics, chemistry, climate changes, biology, population dynamics, engineering, etc. There are also examples where Chaos can be used and/or manipulated for certain purposes. It is the case of chaotic encryption which is used to protect digital information, dynamical chaos control and chaos synchronization.

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