ON THE NUMBER OF LIMIT CYCLES OF CERTAIN POLYNOMIAL SYSTEMS

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ABSTRACT

In this paper, we consider two dimensional autonomous system of the form:

$$\left.\begin{aligned} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \end{aligned}\right] \qquad (A)$$

in which P and Q are polynomials in x and y. We write the system A in the form of

$$\begin{aligned} \dot{x} &= \lambda x + y + p_2(x,y) + p_3(x,y) \\ \dot{y} &= -x + \lambda y + q_2(x,y) + q_3(x,y) \end{aligned} \qquad (B)$$

Where p_2, q_2 and p_3, q_3 are homogeneous quadratic and cubic polynomials in x and y. The question of interest is the maximum possible number of limit cycles (a limit cycle is an isolated closed orbit) of such systems which can bifurcate out of the origin in terms of the degree of P and Q. It is second part of known Hilbert's sixteenth problem. Research on Hilbert's sixteenth problem in general usually proceeds but the investigation of particular classes of polynomial system. In this paper, in particular it is given that up to six limit cycles can bifurcate from fine focus of some examples of cubic system (B). Also we have given one example of quadratic system with at most one limit cycle.

***Keywords**:Limit cycles, perturbation, bifurcation, autonomous system, fine focus.*

INTRODUCTION

In this work we are concerned with the number of limit cycles of two dimensional autonomous systems of the form

$$\left.\begin{aligned} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \end{aligned}\right] \qquad (1.1)$$

where *P* and *Q* are polynomials in *x* and *y*. This problem remains one of the outstanding unsolved problems in the theory of non-linear deferential equations. The maximum possible number of limit cycles (isolated closed orbits) such system have, is the part of sixteenth problem asked by Hilbert in 1900 in International Congress of Mathematicians in Paris [2]. In this work we are mainly concerned with this problem in particular cases.

Limit cycles of plane autonomous differential system appear in the very famous classical paper of H. Poincare [5] and he pioneered the study of the disposition of trajectories of systems of the form (1.1) in their entire domain of existence without integrating the equation. Existence, non-existence and uniqueness of limit cycles have been studied extensively but little has been done on the number of limit cycles of polynomials systems. In 1950, many mathematical models from Physics, Engineering, Chemistry, Biology and Economics etc., were displayed as plane autonomous systems with limit cycles. Also due to the well-known paper of I. G. Petrovski [3] and E. M. Landis [3and 4] concerning the maximum number of limit cycles of all quadratic polynomial systems. This problem has become more important and attracted the attention of many Pure and Applied Mathematicians [3, 6 and 7].

Suppose that the origin is a critical point of the system (1.1) then we can write the system (1.1) as;

$$\left.\begin{aligned} \dot{x} &= ax + by + p(x, y) \\ \dot{y} &= cx + dy + q(x, y) \end{aligned}\right] \qquad (1.2)$$

where p and q contains no linear terms. It is well known that the orbits of the system (1.2) in neighborhood of the origin behave in a similar way to the orbits corresponding to the linear system;

$$\left.\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned}\right] \qquad (1.3)$$

except when

i. $ad - bc = 0$
ii. Origin is a centre or singular node for the system (1.3).

Definition 1*.The origin of the system (1.3) is a centre iff*

$$ad - bc \neq 0 \; and \; a + d = 0$$

Remark 1*.If the origin is a centre for (1.3), the origin of the non-linear system (1.2) could be centre or focus. If it is focus, it is possible to bifurcate limit cycles out of it. Often these bifurcating limit cycles are called local or small amplitude limit cycles.*

Definition 2*. The origin of the system (1.2) is called a fine-focus if it is a centre of linear system (1.3) but non-linear system (1.2).*

When the origin of the system (1.2) is focus or fine focus, we can take canonical coordinates to transform (1.2) to the form:

$$\left.\begin{aligned} \dot{x} &= \lambda x + y + p(x, y) \\ \dot{y} &= -x + \lambda y + q(x, y) \end{aligned}\right] \qquad (1.4)$$

Then origin is a fine focus iff $\lambda = 0$
We are mainly concerned with cubic systems that is, system of the form

$$\left.\begin{aligned} \dot{x} &= \lambda x + y + p_2(x, y) + p_3(x, y) \\ \dot{y} &= -x + \lambda y + q_2(x, y) + q_3(x, y) \end{aligned}\right] \qquad (1.5)$$

where $p_k \; and \; q_k ; k = 2, 3$ are homogeneous polynomial of degree k. The linear part of (1.5) is in canonical form and the stability of the origin is determined by the sign of λ, origin is stable or unstable either $\lambda < 0 \; or \; \lambda > 0. If \; \lambda = 0$, the origin is a centre for the linearized system and is said to be a fine focus (or a weak focus) of the non-linear system. Systems of the form (1.5) were considered in [1, 25 and 36], where they have given certain classes of (1.5) with several small amplitude limit cycles.

When $p_3 = q_3 = 0$ then (1.5) becomes quadratic system have been widely studied; it is known that $H_2 \geq 4$, but no more precise estimate has yet been established and it is still unsolved even whether H_2 is finite. When $p_2 = q_2 = 0$ in equation (1.5) becomes cubic system were investigated in [1 and

41], where it was proved that at most five limit cycles can bifurcate out of the origin. Cubic systems in general (1.1) are less well understood; we referred to [1, 25, 35, 36 and 37].

Our study of cubic system (1.5) is entirely local. We write the equation (1.5) in the form;

$$\left.\begin{aligned} \dot{x} &= \lambda x + y + Ax^2 + (B+2D)xy + Cy^2 + Fx^3 + Gx^2y + (H-3P)xy^2 + Ky^3 \\ \dot{y} &= -x + \lambda y + Dx^2 + (E-2A)xy - Dy^2 + Lx^3 + (M-H-3F)x^2y + (N-G)xy^2 + Py^3 \end{aligned}\right] \quad (1.6)$$

It is the form used in [36]. In [36] they found an example of system of the following form with six small-amplitude limit cycles.

$$\left.\begin{aligned} \dot{x} &= \lambda x + y + Cy^2 + Hxy^2 + Ky^3 \\ \dot{y} &= -x + \lambda y + D(x^2 - y^2) + Lx^3 - Hx^2y + Nxy^2 \end{aligned}\right] \quad (1.7)$$

Other examples of cubic systems with several limit cycles are given by Wang & Luo [38]. Especially the work of Li and co-authors is interesting [26, 27, 28 and 33] and is reported by Tian in [42].

It was proved in [1 and 11] that quadratic system can have at most three small amplitude limit cycles that is in (1.6) and at most five small amplitude limit cycles if the quadratic terms are absent in (1.6). We found one example of quadratic system and prove that this system has at most one limit cycles (given in section 5).

There are many methods to bifurcate limit cycles out of the origin. The one we follow is classical method [1] of seeking Liapunov function *V* in the neighborhood of the origin for which *V*, the rate of change of trajectories is of the form;

$$\eta_2 r^2 + \eta_4 r^4 + \dots,$$

where $\eta_2, \eta_4, \dots$ are constants to be found, known as focal values. We describe a procedure for determining such a function in section *3*. This involves some extremely complicated calculation with polynomials in the coefficients of p_2, q_2 *and* p_3, q_3. We therefore present an algorithm which enables the calculations to be implemented on a computer, exploiting existing techniques for the formal manipulation of polynomials in many variables with integer coefficients. We used a computer algebra package MAPLE for this purpose. The main results are given in section 4.

CENTRE CRITERIA

In the local study of the system (1.1), we find that the problem of a centre is closely related to the problem of number of limit cycles. This problem of centre consists of all obtaining necessary and sufficient conditions that bear on the coefficients of P and Q, in order that all orbits in a neighborhood of the origin is periodic. To calculate number of limit cycles η_k's, we find some value $k_{\max}$ by proving that origin is a centre if $\eta_{2k} = 0$ for $k \le k_{\max} + 1$. Therefore we need conditions which are sufficient for the origin to be a centre [1 and 8]

Theorem 1. *If the origin is a critical point of focus type and*

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0$$

then the origin is a centre for the system (1.1).

Theorem 2. *Symmetry Principle*
Suppose in equation (1.1)

$$P(x, y) = P(-x, y)$$
$$Q(-x, y) = -Q(x, y)$$

and the origin is the only singular point on the y-axis. If a trajectory Γ starts from the positive y-axis and returns to the negative y-axis, then Γ is a closed trajectory. If all the trajectories near the origin posses this property, then origin is a centre.

***Theorem 3**. Symmetry Principle*
Suppose in equation (1.1)

$$P(x,-y) = -P(x,y)$$
$$Q(x,-y) = Q(x,y)$$

and the origin is the only singular point on the x-axis. If the trajectory Γ starts from the positive x-axis and return to the negative x-axis, then Γ is a closed trajectory. If all the trajectories near the origin posses this property, then origin is a centre.

METHOD FOR CALCULATION OF FOCAL VALUES

For completeness we are describing the method here which is given in [1], the procedure for calculation of focal values η_{2k} for the system (1.1) and to determine the Liapunov function *V*. It is convenient to write the system (1.4) in the form;

$$\left.\begin{aligned}\dot{x} &= \lambda x + y + p_2(x,y) + p_3(x,y) + \ldots + p_n(x,y)\\ \dot{y} &= -x + \lambda y + q_2(x,y) + q_3(x,y) + \ldots + q_n(x,y)\end{aligned}\right] \tag{3.1}$$

where, for *k=2,3,…,n,* p_k *and* q_k are homogeneous polynomials of degree *k*. We take a function *V*

$$V(x,y) = \frac{1}{2}\left(x^2+y^2\right) + V_3(x,y) + \ldots V_k(x,y) + \ldots$$

for $k \ge 3$ are homogeneous polynomial of degree k. Further we can write

$$V_k = \sum_{i=0}^{k} V_{k-i,i} x^{k-i} y^i$$

and V_k for the column vector $\left(V_{k,0}, V_{k-1,1}, \ldots, V_{0,k}\right)^T$.

The rate of change of *V* along a trajectory is $\dot{V}$ such that:

$$\left.\begin{aligned}\dot{V} = &\left(x + (V_3)_x + (V_4)_x + \ldots\right)\left(\lambda x + y + p_2(x,y) + \ldots + p_n(x,y)\right)\\ &+\left(y + (V_3)_y + (V_4)_y + \ldots\right)\left(-x + \lambda y + q_2(x,y) + \ldots + q_n(x,y)\right)\end{aligned}\right] \tag{3.2}$$

Where suffices *x* and *y* denote partial differentiation with respect to *x* and *y* respectively.

We write D_k for the collection of terms of degree *k* on right hand side of (3.2). Clearly for $k \ge 3$,

$$D_k = \left[y(V_k)_x - x(V_k)_y\right] + \left[(V_{k-1})_x p_2 + (V_{k-1})_y q_2 + \ldots + xp_{k-1} + yq_{k-1}\right] \tag{3.3}$$

The polynomial in the second parenthesis has coefficients which can be expressed linearly in terms of the coefficients in p_k *and* q_k.

Here we chose coefficients $V_{i,j}$ and quantities η_k so tha

i. $D_k = 0$ if k is odd and

ii. $D_k = \eta_k \left(x^2+y^2\right)^{\frac{1}{2k}}$ if *k is even.*

For convenience, we say that $V_{i,k-1}$ is an odd or even coefficient of V_k according to whether *i* is odd or even.

Suppose first that *k* is odd. The requirement $D_k = 0$ is equivalent to a set of *k+1* linear equations for *k+1* unknowns $V_{k,0}, V_{k-1,1}, \ldots, V_{0,k}$. These uncouple into two sets of *(k+ 1)/2* equations, one set determines the odd coefficients of V_k and the other determines the even coefficients and V_k is uniquely determined.

When k is even (*k= 2m,* say) the condition $D_k = 0$ is equivalent to *k+1* linear equations which again uncouple into two sets: *m+1* equations involving the *m* odd coefficients of V_k and *m* equations involving the *m+ 1* even coefficients. Thus we can not arrange that $D_k = 0$, due to which we introduce the new variable η_k, and require $D_k = \eta_k r^k$ for which we have *m* equations.

When *k* is a multiple of 4, we impose the extra condition

$$V_{m,m} = 0$$

When *k= 2(mod4)* there is no "middle" term to set equal to zero, instead we split the middle equation namely;

$$(m+1)V_{m+1,m} + (m+1)V_{m,m+1} = 0$$

In summarize for any class of given system, there are four steps to proceed

i. Calculation of the focal values.
ii. Reduction of the focal values to obtain the Liapunov quantities.
iii. Establishing the value of $k_{\max}$ by proving that the origin is a centre if $\eta_{2k} = 0 \; for \, k \le k_{\max} + 1$.
iv. Commencing with a fine focus of maximal order, finding a sequence of perturbation each of which reverse the stability of the origin.

METHOD OF BIFURCATING LIMIT CYCLES FROM THE ORIGIN

Much of the recent progress on Hilbert's sixteenth problem has involved the construction of systems with small amplitude limit cycles. These are the limit cycles which bifurcate out of a critical point under perturbation of the equations. Now question is how these small-amplitude limit cycles can be generated. There are various ways of establishing the existence of limit cycles. These may fall into two main categories: (*i*) Local (*ii*) global. In **local** methods analysis is restricted to a small neighborhood of a critical point of an orbit. Whereas methods in which analysis require studying orbits in the large, are named as "**global**".

There are two main methods of proving the existence of limit cycles. The first one depends on the following form the Poincare Bendixson theorem:

Theorem 4. *If the system*

$$\dot{x} = P(x, y)$$

$$\dot{y} = Q(x, y)$$

has a solution which remains in a bounded region and does not approach an equilibrium point then the solution is either itself a periodic orbit or it spirals towards a solution which is periodic.

So it is easy to prove the existence of a limit cycle by determining a bounded region in which trajectories remain and which does not contain a critical point. Second method is bifurcation techniques. The phenomenon related to the exchange of stability of solution is called bifurcation when parameters or coefficients are perturbed.

In this section we are describing how these small amplitude limit cycles can be bifurcated when we perturbed the coefficients of the system with origin as a fine focus.

The idea is to start with a system (1.4) for which the origin is a critical point of focus type whose order is as large as possible, and to make a sequence of perturbations of the coefficients occurring in P and Q, each of which reserves the stability of the origin, thereby generating a small amplitude limit cycles.

Let τ be the collection of two dimensional autonomous differential systems of the form (1.1) and let the system $S \in \tau$ has a fine focus of order k. Then by definition $L(0) = \ldots = L(k-1) = 0,$ and $L(k) \neq 0$ for *S*. Without loss of generality, we take $L(k) < 0,$ so that origin is stable. Let Γ be a level curve of *V* which is sufficiently near the origin so the flow is inward across it. Now as we perturbed S, we get $S_1 \in \tau,$ so that

$$L(0) = L(1) = \ldots = L(k-2) = 0$$

But $L(k-1) > 0,$ the origin for S_1 is unstable. Now we take Liapunov function V_1 corresponding to S_1, let us take level curve Γ_1 of V_1 inside Γ and sufficiently near the origin so that the flow is outward across Γ_1 By the Poincare-Bendixson theorem, there is a limit cycle between Γ and Γ_1.

The next step is to take a perturbation of S_2 of S_1 so that

$$L(0) = L(1) = \ldots = L(k-3) = 0$$

and $L(k-2) < 0;$ if the perturbation is sufficiently small, the flow remains inward across Γ and outward Γ_1, whence S_2 has a limit cycle between Γ and Γ_1 (See fig.). As the origin is stable for S_2, there is also a limit cycle

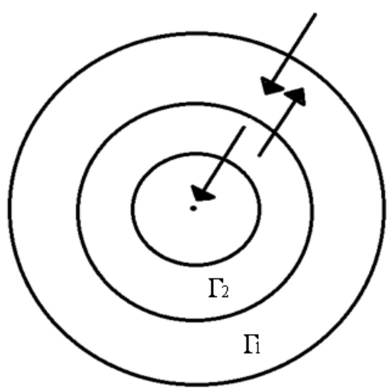

Figure: Limit cycles

inside Γ_1. By continuing in this way, we can produce *k* number of limit cycles. The required conditions are:

$$L(j)L(j+1) < 0, |L(j)| << |L(j+1)| \quad (j = 0,1,\ldots,k) \tag{4.1}$$

This strategy may fail for one of the following two reasons.

i. It may be that less than *k* Liapunov quantities are derived from $\eta_2, \ldots, \eta_{2k+2}$.
ii. It may happen from that the condition (4.1) can not be satisfied by perturbations which remain within the class of equations under consideration.

We have found an example which has less Liapunove quantities.

1. Some Examples

We are giving here firstly the example of quadratic system with at most one limit cycle. Consider

$$\dot{x} = \lambda x + y + ax^2 + (b+2d)xy + (c-a)y^2$$
$$\dot{y} = -x + \lambda y + dx^2 + 2axy - dy^2 \qquad (5.1)$$

Using the algorith in explained in section 3, we have $\eta_2 = \lambda$. So $L(0) = \lambda$ and now we calculate η_4. For this we substitute $\lambda = 0$ in (5.1). Here we put $V_{2,2} = 0 \, as \, k = 4$, we get

$$\eta_4 = -\frac{1}{8}bc$$

Here $L(1) = -bc$. For a fine focus of order greater than one we take $L(1) = 0$, that $bc = 0$. So either $b = 0 \, or \, c = 0$.

1. If $b = 0$, $c \neq 0$ then the system becomes;

$$\dot{x} = \lambda x + y + ax^2 + 2dxy + (c-a)y^2$$
$$\dot{y} = -x + \lambda y + dx^2 - 2axy - dy^2$$

and

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0$$

Thus origin is centre by theorem 1.

2. Now we take $b \neq 0, c = 0$ then the system takes the form;

$$\dot{x} = \lambda x + y + ax^2 + (b+2d)xy + ay^2$$
$$\dot{y} = -x + \lambda y + dx^2 - 2axy - dy^2$$

We have calculated $\eta_6, \eta_8 \, and \, \eta_{10}$ and found $\eta_6 = \eta_8 = \eta_{10} = 0$. So this is an example of one limit cycle.

Theorem 5. For the system (5.1)

$$L(0) = \lambda$$
$$L(1) = -\frac{1}{8}bc$$

If we choose $c > 0 \, and \, b > 0 \, then \, L(1) < 0$, choosing $\lambda > 0$ we have $L(0)L(1) < 0$. So we have one limit cycle.

Example 1. Now we are going to consider the simple classes of system (B);

$$\dot{x} = \lambda x + y + ax^2 + cx^3$$
$$\dot{y} = -x + \lambda y + dy^3 + exy^2 \qquad (5.2.1)$$

Using the algorithm explained in section 3, we have $L(0) = \eta_2 = \lambda$. We set $\lambda = 0$. Then we calculate

$$\eta_4 = \frac{3}{8}(c+d)$$

Thus

$$L(1) = (c+d)$$

For a fine focus of order greater than one, we take $L(1)=0$, so that $(c+d)=0$. By taking

$$c=-d \quad \text{(C)}$$

The system (5.2.1) becomes

$$\begin{aligned}\dot{x} &= y+ax^2-dx^3\\ \dot{y} &= -x+dy^3+exy^2\end{aligned} \quad (5.2.2)$$

Then we calculate

$$\eta_6=-\frac{5}{48}da^2+\frac{1}{16}de$$

So the Liapunov quantity can be written as

$$L(2)=\eta_6=-d\left(5a^2-3e\right)$$

Then for the fine focus of order greater than two, we take $\eta_6=0, \mathit{so\,that}\; d\left(5a^2-3e\right)=0.$ Then we have two cases:

1. $d=0, or$
2. $5a^2-3e=0$

Case 1. If we have d = 0, then equation (C) gives us c= 0 and hence by theorem 1, the origin is centre. So $d\neq 0$.

Case 2. Then we take $5a^2-3e=0,$ so

$$e=\frac{5}{3}a^2 \quad \text{(D)}$$

Then the system (5.2.2) takes the form

$$\begin{aligned}\dot{x} &= y+ax^2-dx^3\\ \dot{y} &= -x+dy^3+\frac{5}{3}a^2xy^2\end{aligned} \quad (5.2.3)$$

We then calculate

$$\eta_8=\frac{5}{96}a^4d$$

so that

$$L(3)=a^4d$$

and

$$L(0)=\lambda, L(1)=(c+d), L(2)=-d\left(5a^2-3e\right), L(3)=a^4d$$

We therefore have the following lemma.

Lemma 1. The origin of the system (5.2.1) is a fine focus of order three if $\lambda=0, c=-d$ and $5a^2-3e=0$ are satisfied and if $ad\neq 0.$ As if we take $d=0,$ then the origin is centre. So $d\neq 0, \mathit{and\,if}\; a=0,$ then all the Liapunov quantities $L(4), L(5),\ldots$ are zero.

Remark 2. If $a=0,$ then we conjecture that origin is a centre for

$$\left.\begin{aligned}\dot{x} &= y-cx^3\\ \dot{y} &= -x+dy^3\end{aligned}\right] \quad \text{(E)}$$

Theorem 6. Suppose that the origin of the system (5.2.1) is a fine focus of order three (i.e. it satisfies the condition of Lemma 1). Then by suitable perturbation of the coefficients three limit cycles can be bifurcated out of the origin.

We define:

$$\alpha = c + d$$
$$\beta = 5a^2 - 3e$$
$$\gamma = a^4 d$$

Since origin is a fine focus of order three, therefore by Lemma 1;

$$\lambda = \alpha = \beta = 0 \, but \, \gamma \neq 0$$

That is $L(k) = 0 \; for \, k < 3 \, but \, L(3) \neq 0.$ For definiteness, we suppose that d > 0, then $L(3) > 0.$ First we perturb *e* so that $L(k) = 0$, for $k \le 1$ but $L(3)L(2) < 0.$ This is achieved by decreasing *e*. At the same time we adjust the values of *c*, so that $\alpha = 0.$ Thus the order of the fine focus is reduced by one and the stability of the origin is reversed. Thus a limit cycle bifurcates out of the origin.

Secondaly, we perturb *c* so that $L(1) > 0$, which is achieved by increasing *c,* then $L(2)L(1) < 0$ *and* $L(k) = 0$ for *k < 1*.There is again reversal of the stability at the origin and a limit cycle bifurcate out of the origin. If the perturbation is small enough, the first limit cycle persist and we have two limit cycles. Next take $\lambda < 0,$ so that $L(1)L(0) < 0.$ Thus the stability of the origin is reversed and a limit cycle bifurcate out of the origin. So we have three limit cycles if the perturbation is small enough.

Similarly if we take $\dot{x}$ without quadratic term i.e. $a = 0, \; in(5.2.1)$ then we have following theorem.

Theorem 7. For the system (5.2.1) with $a = 0$ we have

$$L(0) = \lambda$$
$$L(1) = (c + d)$$
$$L(2) = \frac{5}{48} db^2 + \frac{1}{16} de$$
$$L(3) = b^4 d$$

and the system (5.2.1) is fine focus of order at most three.

2. General Cubic System

In this section we consider some example of following type:

$$\left.\begin{aligned} \dot{x} &= \lambda x + y + ax^2 + (b+2d)xy + (c-a)y^2 + fx^3 + gx^2y + (l-3u)xy^2 + sy^3 \\ \dot{y} &= -x + \lambda y + dx^2 + (e-2a)xy - dy^2 + ex^3 + (p-l-3f)x^2y + (q-g)xy^2 + uy^3 \end{aligned}\right] \quad (6.1)$$

This is the form used in [35], we put $c - a$ instead of *c*. Using the *algorithm 3* and computer program, we get the simplified form of Liapunov quantities

$$L(0) = \lambda$$
$$L(1) = bc - p$$

$$L(2)=\frac{3}{4}dab^2+\frac{5}{48}de^2-\frac{1}{16}ep-\frac{5}{8}ef-\frac{1}{16}le+\frac{1}{8}ecb+\frac{1}{4}ab^3-\frac{11}{48}fb^2-\frac{1}{4}fg-\frac{2}{3}dfb$$
$$-\frac{1}{6}d^2cb+\frac{9}{16}eab+\frac{1}{2}fae-\frac{1}{12}aue-\frac{7}{48}ac^2b-\frac{5}{24}acp+\frac{7}{24}abg-\frac{1}{32}aep$$
$$+\frac{13}{48}asb+\frac{1}{12}ale--\frac{1}{6}alc-\frac{1}{6}a^2cb-\frac{35}{96}aecb+\frac{25}{192}c^2p-\frac{15}{64}c^3b-\frac{1}{16}sp+\frac{1}{8}uq$$
$$+\frac{1}{48}ue^2-\frac{1}{192}cb^3+\frac{5}{8}c^2f+\frac{5}{48}c^2l+\frac{1}{24}esb+\frac{11}{48}scb-\frac{3}{32}dcb^2-\frac{1}{32}dbp+\frac{1}{24}\deg$$
$$-\frac{1}{48}deq+\frac{1}{12}dlb-\frac{1}{4}dbu-\frac{5}{48}cue+\frac{1}{6}cdq+\frac{1}{48}cbg+\frac{11}{96}cep+\frac{7}{16}cef+\frac{1}{12}cle$$
$$-\frac{19}{96}c^2eb-\frac{5}{48}c^2de+\frac{1}{24}be^2-\frac{1}{64}b^2p-\frac{1}{16}b^2u+\frac{1}{48}b^2l+\frac{1}{64}e^2p-\frac{1}{16}e^2f$$
$$-\frac{1}{48}e^2l-\frac{3}{8}sf-\frac{1}{32}qp-\frac{1}{4}qf-\frac{1}{16}ls-\frac{1}{4}ba^2e+\frac{5}{48}baq+\frac{1}{192}e^2cb+\frac{1}{48}e^2dc$$
$$+\frac{1}{16}des+\frac{3}{32}qcb-\frac{1}{16}lq$$

Using $p=bc$, we have,

$$L(2)=-\frac{5}{8}ef-\frac{1}{16}le+\frac{1}{4}ab^3-\frac{11}{48}fb^2-\frac{1}{4}fg-\frac{5}{48}c^2b+\frac{1}{8}uq+\frac{1}{48}ue^2-\frac{1}{48}cb^3$$
$$+\frac{5}{8}c^2f+\frac{5}{48}c^2l+\frac{1}{24}e^2b-\frac{1}{16}b^2u+\frac{1}{48}b^2l-\frac{1}{16}e^2f-\frac{1}{48}e^2l-\frac{3}{8}sf$$
$$-\frac{1}{4}qf-\frac{1}{16}ls+\frac{5}{48}de^2-\frac{1}{16}lq+\frac{1}{6}scb-\frac{1}{8}dcb^2+\frac{1}{24}\deg-\frac{1}{48}deq+\frac{1}{12}dlb$$
$$-\frac{1}{4}dbu-\frac{5}{48}cue+\frac{1}{6}cdq+\frac{1}{48}cbg+\frac{7}{16}cef+\frac{1}{12}cle-\frac{1}{12}c^2eb-\frac{5}{48}c^2de$$
$$-\frac{1}{4}ba^2e-\frac{5}{48}baq^3+\frac{1}{48}e^2bc+\frac{1}{48}e^2dc+\frac{1}{16}des+\frac{1}{16}qcb+\frac{3}{4}dab^2+\frac{1}{16}ecb$$
$$-\frac{2}{3}dfb-\frac{1}{6}d^2cb+\frac{9}{16}eab+\frac{1}{2}fae-\frac{1}{12}aue-\frac{17}{48}ac^2b+\frac{7}{24}abg+\frac{13}{48}asb$$
$$+\frac{1}{12}ale-\frac{1}{6}alc-\frac{1}{6}a^2cb+\frac{1}{24}esb-\frac{19}{48}aecb$$

Still it contains too many terms. To make progress, we clearly must consider particular cases of (6.1), i.e. we put

$e=0, b=0, c=0, q=0, g=0, f=0, s=0$

Then we get a following example.

Example 1.

$$\dot{x}=y+ax^2+2dxy-ay^2+(l-3u)xy^2+sy^3$$
$$\dot{y}=-x+dx^2-2axy-dy^2-lx^2y+uy^3 \quad (6.2)$$

Using algorithm defined in section 3 and computer program, we get the simplified form of Liapounov quantities;

$L(0)=\lambda$

$L(1)=bc-p$

$L(2)=-sl$

So $L(2)=-sl.$ For the fine focus of order greater than two, we take $L(2)=0$, so that $sl=0$ it implies that $l=0\, or\, s=0.$

Case 1: If we take $l=0,$ then by theorem 1, origin is centre. So $l\neq 0.$

Case 2: If we take

$s=0$ (c)

then the system in (6.2) takes the form where as

$$\dot{x}=y+ax^2+2dxy-ay^2+(l-3u)xy^2$$
$$\dot{y}=-x+dx^2-2axy-dy^2-lx^2y+uy^3 \qquad (6.3)$$

Using the algorithm described in section 3 and computer program, we get the Liapunov quantities of the form

$$\eta_8=\frac{35}{48}adlu-\frac{1}{32}l^2u+\frac{5}{64}u^2l$$

So $L(3)=lu(140ad-6l+15u)$

For a fine focus of order greater than three, we take $L(3)=0, so$

$lu(140ad-6l+15u)=0$

Hence we have two choices:

1. $u=0, l\neq 0, 140ad-6l+15u\neq 0$
2. $140ad-6l+15u=0, u=0, l\neq 0$

Case 1: If we take

$u=0, l\neq 0$ (D)

Then the system (6.3) takes the form;

$$\dot{x}=y+ax^2+2dxy-ay^2+lxy^2$$
$$\dot{y}=-x+dx^2-2axy-dy^2-lx^2y \qquad (6.4)$$

Using the algorithm described in section 3 and computer program, we get following form of Liapunov quantities,

$$\eta_{10}=-\frac{1}{16}a^2l^3+\frac{1}{16}d^2l^3$$

So

$L(4)=-l^3(a-d)(a+d)$

For a fine focus of order greater than four, we take $L(4)=0=-l^3(a-d)(a+d).$ So there we have two choices as $l\neq 0$

1. $a-d=0, a+d\neq 0$
2. $a+d=0, a-d\neq 0$

Case 1(i): if $a-d=0, as\, l\neq 0$ that is

$d = a$ (E)

Then the system (6.4) takes the form

$$\dot{x} = y + ax^2 + 2axy - ay^2 + lxy^2$$
$$\dot{y} = -x + ax^2 - 2axy - ay^2 - lx^2 y \quad (6.5)$$

Using the algorithm described in section 3 and computer programe, we get $\eta_{12} = 0, so\ L(5)$ is derived from $\eta_{14}\ and\ L(6)$ is derived from η_{16} as under;

$$L(5) = a^2 l\left(7776160a^6 l + 32880a^4 l^2 + 137715200a^8 - 2097333a^2 l^3 + 2097333l^5 - 32880a^2 l^4\right)$$

$$L(6) = -a^2(-195094908a^4 l^3 - 2163915a^2 l^4 + 169384608a^2 l^5 + 12440144640a^6 l^2 - 24286787840a^8 l + 32222400a^4 l^4 + 362333440000a^{10} + 2163915l^6)$$

So the system (6.2) is a fine focus of the order seven.

Case 1(ii): If $a + d = 0$ that is

$d = -a$ (F)

Then the system (6.3) takes the form

$$\dot{x} = y + ax^2 - 2axy - ay^2 + lxy^2$$
$$\dot{y} = -x - ax^2 - 2axy + ay^2 - lx^2 y \quad (6.6)$$

We get all next Liapunov quntities are zero. i.e. $\eta_{12} = \eta_{14} = 0$

Case2: If $140ad - 6l + 15u = 0, then$

$$u = \frac{1}{15}(6l - 140d) \quad \text{(G)}$$

And the system (6.3) takes the form

$$\dot{x} = y + ax^2 + 2dxy - ay^2 + (l - \frac{1}{5}(6l - 140d))xy^2$$
$$\dot{y} = -x + dx^2 - 2ax - dy^2 - lx^2 y + \frac{1}{15}(6l - 140d)y^3 \quad (6.7)$$

For the system (6.7), we now calculate η_{10}, and we have

$$\eta_{10} = \frac{169}{800}l^3 d^2 - \frac{931}{48}ad^3 l^2 - \frac{473}{720}a^3 dl^2 - \frac{109}{800}a^2 l^3 + \frac{4459}{12}a^2 d^4 l + \frac{20923}{108}a^4 d^2 l$$

So that

$$L(4) = \frac{1}{21600}l\left(4563l^2 d^3 - 418950ad^3 l - 142170a^3 dl - 2943l^2 a^3 + 8026200a^2 d^4 + 4184600a^4 d^2\right)$$

and the system (6.2) is fine focus of order four.

Theorem 8. For the system (6.2)

$L(0)=\lambda$

$L(1)=bc-p$

$L(2)=-sl$

$L(3)=lu(140ad-6l+15u)$

$L(4)=-l^3(a-d)(a+d)$

$L(5)=a^2l(7776160a^6l+32880a^4l^2+137715200a^8-2097333a^2l^3+2097333l^5-32880a^2l^4)$

$$L(6)=-a^2l(-195094908a^4l^3-2163915a^2l^4+169384608a^2l^5+12440144640a^6l^2$$
$$-24286787840a^8l+32222400a^4l^4+362333440000a^{10}+2163915l^6)$$

For k = 0,12,3,4,5,6, L (k) was derived from η_{2k+2}.

Remark 3. In this case L (6) is derived from η_{14}.

Lemma 2. The origin of the system (6.2) is a fine focus of order seven if $\lambda=0$ and $p=bc$, and conditions (C) to (E) are satisfied.

Theorem 9. For the system (5.7)

$$L(0)=\lambda$$
$$L(1)=bc-p$$
$$L(2)=-sl$$
$$L(3)=lu(140ad-6l+15u)$$
$$L(4)=-l^3(a-d)(a+d)$$

For k = 0,1,2,3,4,5,6, L (k) was derived from η_{2k+2}.

Lemma 3. The origin of the system (6.2) is a fine focus of order five if $\lambda=0$ and $p=bc, s=0, and\ d=-a$ are satisfied.

Theorem 10. For the system (6.2)

$L(0)=\lambda$

$L(1)=bc-p$

$L(2)=-sl$

$L(3)=lu(140ad-6l+15u)$

$L(4)=\frac{1}{21600}l(4563l^2d^2-418950ad^3l-142170a^3dl-2943l^2a^2+8026200a^2d^4+4184600a^4d^2)$

For k = 0,1,2,3,4,5,6, L (k) was derived from η_{2k+2}.

Lemma 4. The origin of the system (6.2) is a fine focus of order five if $\lambda=0$ and $p=bc, s=0, and\ d=a$ are satisfied.

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