NEW DERIVATIVE FREE ITERATIVE METHOD FOR SOLVING NON-LINEAR EQUATIONS

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ABSTRACT

Several iterative methods have been proposed and analyzed in the literature for solving non-linear equation, $f(x) = 0$. *Recently Wu et al have suggested derivative free method for solving non-linear equations. Other well-known methods with derivatives create numerical difficulties or fail to converge in neighborhood of the required root. In this paper, we propose and analyze two two-step derivative free algorithms. The numerical tests show that the new two-step algorithms are comparable with the existing algorithms and are successful in case where the existing algorithms fail to converge or have numerical difficulties.*

Keywords: *Nonlinear equations, Iterative methods, Two-step methods, Derivative free methods, Numerical examples.*

INTRODUCTION

Finding the roots of a single variable nonlinear equations efficiently, is a very basic and old problem in numerical analysis and has many applications in applied sciences.

In recent years, researchers have developed many iterative methods for solving one variable non-linear equations. These methods are developed using error analysis techniques, quadrature rules or other techniques, see [1 – 5] and the reference therein. Recently, Ujevic [7] and Noor, et al [6] have suggested two-step methods for solving non-linear equations. These two-step methods include methods, with derivatives in their denominator, which create numerical difficulties.

Ujevic [7] and Wu et al [8] have suggested derivative free iterative methods for solving non-linear equations

$$f(x) = 0.$$

Inspired and motivated by the research going on in this direction, we propose and analyze two two-step iterative methods by combining Ujevic [7] and Wu and Fu [8] methods for solving non-linear equations. Numerical comparison between the existing **Algorithms** and the newly developed algorithms is given. Numerical comparison shows that the new two-step algorithms are comparable with the existing algorithms and are successful in case where the existing algorithms fail to converge or have numerical difficulties. Our results can be considered as an improvement and refinement of the previously known results in the literature.

TWO-STEP ITERATIVE METHODS

First, we describe the already existing two-step iterative methods in the literature.

Algorithm *NW* ***(Classical Newton's Method**):*

Step 1. *For given* x_o, *calculate* $x_1, x_2, \quad \cdots \quad,$ *such that*

$$z_k = x_k - \frac{f(x_k)}{f'(x_k)},$$

Step 2: *For given* $\varepsilon > 0,$ *if* $|x_{k+1} - x_k| < \varepsilon,$ *then stop.*

Step 3: *Set* $k = k+1$ *and go to* ***Step 1****.*

The following two-step method is due to Ujevic [7] . This method is a combination of a method derived using special type of quadrature rule and Newton's method with a parameter $\alpha \in (0,1]$ and is described as:

Algorithm ***(due to Ujevic** [7]):*

Step 1. *For given* x_o, *calculate* $x_1, x_2, \quad \cdots \quad,$ *such that*

$$z_k = x_k - \alpha\frac{f(x_k)}{f'(x_k)},$$

$$x_{k+1} = x_k + 4(z_k - x_k)\frac{f(x_k)}{3f(x_k) - 2f(z_k)}, \; k = 0,1, \ldots.,$$

where $\alpha \in (0,1]$ *is known as controlling parameter.*

Step 2: *For given* $\varepsilon > 0,$ *if* $|x_{k+1} - x_k| < \varepsilon,$ *then stop.*

Step 3: *Set* $k = k+1$ *and go to* ***Step 1****.*

The following method is due to Wu et al [8] .

Algorithm *WU(due to Wu et al* ***[8]):***

Step 1. *For given* x_o, *calculate* $x_1, x_2, \quad \cdots \quad,$ *such that*

$$x_{k+1} = x_k - \frac{f^2(x_k)}{pf^2(x_k) + f(x_k) - f(x_k - f(x_k))},$$

where sign of p *should be taken so as to make the denominator largest in magnitude. Here* $p \in \mathbb{R}$ *is chosen, so that* $f(x_k) - f(x_k - f(x_k))$ *and* p *have the same sign.*

Step 2: *For given* $\varepsilon > 0,$ *if* $|x_{k+1} - x_k| < \varepsilon$ *, then stop.*

Step 3: *Set* $k = k+1$ *and go to* ***Step 1****.*

We now suggest new two-step iterative methods by using one-step quadratically convergent derivative free methods by Wu et al and Ujevic method as corrector, once with controlling parameter q in the numerator and secondly, as controlling parameter p in the denominator. The algorithm is described as follows:

Algorithm *ANF:*

Step 1: *For initial guess* x_o, *a tolerance* $\varepsilon > 0$, *a controlling parameter* $q > 0$ *and for iterations* n , *set* $k = 0$,

if

$$f(x_k) - f(x_k - f(x_k)) > 0,$$

then,

$p = 1,$

else

$p = -1,$

Step 2: *Calculate*

$$z_k = x_k - \frac{qf^2(x_k)}{pf^2(x_k) + f(x_k) - f(x_k - f(x_k))},$$

$$x_{k+1} = x_k + 4(z_k - x_k)\frac{f(x_k)}{3f(x_k) - 2f(z_k)},$$

where sign of p *should be taken so as to make the denominator largest in magnitude. Here* $p \in \mathbb{R}$ *is chosen, so that* $f(x_k) - f(x_k - f(x_k))$ *and* p *have the same sign.*

Step 3: *If* $|x_{k+1} - x_k| < \varepsilon$ *or* $k > n$ *then stop.*

Step 4: *Set* $k \longrightarrow k+1$ *and go to* ***Step 1.***

CONVERGENCE ANALYSIS

Now, we prove that our two-step method (ANF) has second order of convergence.

Theorem: *Let* $\alpha \in I$ *be a simple zero of a sufficiently differentiable function* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *on an open interval* I. *If* x_0 *is close to* α, *then the algorithm* 4 *has second order of convergence.*

Proof: The iterative scheme is given by

$$z_n = x_n - \frac{qf^2(x_n)}{pf^2(x_n) + f(x_n) + f(x_n - f(x_n))},$$

$$x_{n+1} = x_n + 4(z_n - x_n)\frac{f(x_n)}{3f(x_n) - 2f(z_n)}.$$

Let α be a simple zero of f . By Taylor's series, we have

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + O(e_n^5)],$$

where

$$c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)},\ k = 2,3,\ \ldots,\ \text{and } e_n = x_n - \alpha.$$

Using 3.3 and by Taylor's series, we have

$$f(x_n - f(x_n)) = f'(\alpha)[-c_2 e_n^2 - c_3 e_n^3 + (-c_4 + c_2^3)e_n^4 + O(e_n^5)].$$

From (3.1), (3.2) and 3.4, we get

$$z_n = \alpha + (1-q)e_n + qpe_n^2 + (-qc_2^2 - qp^2)e_n^3 + O(e_n^4).$$

Also by Taylor series, we obtain

$$\begin{aligned} f(z_n) = f'(\alpha)[(1-q)e_n + (qp + c_2 - 2qc_2 + c_2 q^2)e_n^2 + (-qc_2^2 \\ - qp^2 + 2qpc_2 - 2q^2pc_2 + c_3 - 3qc_3 + 3c_3q^2 \\ - c_3q^3)e_n^3 + O(e_n^4). \end{aligned}$$

Finally using (3.2), (3.3), (3.5) and (3.6), we obtain

$$\begin{aligned} x_{n+1} = \alpha + (1 - \frac{4q}{1+2q})e_n + (-\frac{4qc_2}{1+2q} + \frac{4qp}{1+2q} + \frac{4qc_2}{(1+2q)^2} \\ - \frac{8q^2p}{(1+2q)^2} + \frac{16q^2c_2}{(1+2q)^2} - \frac{8q^3c_2}{(1+2q)^2})e_n^2 + (-\frac{4qc_3}{1+2q} \\ - \frac{4qc_2^2}{1+2q} - \frac{4qp^2}{1+2q} + \frac{4qpc_2}{1+2q} + \frac{4qc_3}{(1+2q)^2} + \frac{8q^2c_2^2}{(1+2q)^2} \\ + \frac{8q^2p^2}{(1+2q)^2} - \frac{16q^2pc_2}{(1+2q)^2} + \frac{16q^3pc_2}{(1+2q)^2} + \frac{24q^2c_3}{(1+2q)^2} \\ - \frac{24q^3c_3}{(1+2q)^2} + \frac{8q^4c_3}{(1+2q)^2} - \frac{8q^2c_2^2}{(1+2q)^3} + \frac{24q^3pc_2}{(1+2q)^3} \\ - \frac{24q^3c_2^2}{(1+2q)^3} + \frac{48q^4c_2^2}{(1+2q)^3} - \frac{4qpc_2}{(1+2q)^3} + \frac{8q^2p^2}{(1+2q)^3} \\ - \frac{16q^2pc_2}{(1+2q)^3} - \frac{16q^4pc_2}{(1+2q)^3} - \frac{16q^5c_2^2}{(1+2q)^3})e_n^3 + O(e_n^4), \end{aligned}$$

it implies that if $q = \frac{1}{2}$, then the above algorithm has second order of convergence,

$$e_{n+1} = (\frac{1}{2}p + \frac{1}{4}c_2)e_n^2 + (\frac{3}{8}c_3 - \frac{13}{16}c_2^2 - \frac{1}{4}p^2)e_n^3 + O(e_n^4).$$

Hence proved.

Remark *With the choice of* $q = \frac{1}{2}$, *the new proposed algorithm 4 takes the following form*

Algorithm ***Step 1:*** *For initial guess* x_o, *a tolerance* $\varepsilon > 0$, *a controlling parameter* $q > 0$ *and for iterations* n , *set* $k = 0$,

if

$$f(x_k) - f(x_k - f(x_k)) > 0,$$

then,

$p = 1,$

else

$p = -1,$

Step 2: *Calculate*

$$z_k = x_k - \frac{\frac{1}{2}f^2(x_k)}{pf^2(x_k) + f(x_k) - f(x_k - f(x_k))},$$

$$x_{k+1} = x_k + 4(z_k - x_k)\frac{f(x_k)}{3f(x_k) - 2f(z_k)},$$

where sign of p *should be taken so as to make the denominator largest in magnitude. Here* $p \in \mathbb{R}$ *is chosen, so that* $f(x_k) - f(x_k - f(x_k))$ *and* p *have the same sign.*

Step 3: *If* $|x_{k+1} - x_k| < \varepsilon$ *or* $k > n$ *then stop.*

Step 4: *Set* $k \rightarrow k+1$ *and go to* ***Step 1.***

NUMERICAL EXAMPLES

Here, we perform some numerical tests and compare the our new proposed method (ANF) with Classical Newton's method (NW), Wu et al's method (WU) [8]. We consider here $\varepsilon = 1.0e-15$ as given tolerance, $q = \frac{1}{2}$ as best choices for controlling parameters for new method (ANF) and $n = 500$ the maximum number of iterations to be performed. Numerical results have been computed using Maple 9 and are provided in the Table **4.1**.

CONCLUSION

It is to be noted that the function and derivative evaluations for all the algorithms are the same. We have proposed new two-step derivative free algorithms for solving single variable non-linear equation. On the basis of numerical tests and comparison between the algorithms, it follows that our algorithms are comparable with the existing methods and are successful.

Table of Functions

	Functions	Root
f_1	$x^2 - 10\cos x,$	-1.379364594222203083
f_2	$x^2 - e^x - 3x + 2,$	0.257530285439860
f_3	$x^3 - 10,$	2.15443469003188372
f_4	$cos(x) - x,$	0.739085133215160
f_5	$(x-1)^3 - 1,$	2
f_6	$sin^2(x) - x^2 + 1,$	1.40449164821534123
f_7	$e^{x^2+7x-30} - 1,$	3

	IT	$f(x_n)$	δ
$f_1,\ x_0 = -1.5$			
NW	5	6.4e-56	1.8e-28
WU	5	8.8e-36	4.7e-19
ANF	5	3.9e-42	4.4e-22
$f_2,\ x_0 = .5$			
NW	5	3.1e-54	3.0e-27
WU	6	-5.9e-49	5.3e-25
ANF	5	-6.1e-34	2.5e-17
$f_3,\ x_0 = 2.5$			
NW	6	4.9e-53	2.7e-27
WU	7	-7.4e-29	8.7e-16
ANF	6	-1.1e-40	1.5e-21
$f_4,\ x_0 = 0.2$			
NW	6	-3.8e-52	3.2e-26
WU	6	8.2e-55	1.0e-27
ANF	5	1.2e-52	2.2e-26
$f_5,\ x_0 = 1.5$			
NW	8	3.9e-45	3.6e-23
WU	8	-2.7e-36	5.5e-19
ANF	7	-7.5e-56	1.4e-28
$f_6,\ x_0 = 1.1$			
NW	6	-1.0e-34	7.3e-18
WU	8	-7.0e-46	8.7e-24
ANF	7	-5.6e-53	3.3e-27
$f_7,\ x_0 = 2$			
NW	Diverge	-----	------
WU	Diverge	------	------
ANF	5	-8.6e-37	4.1e-20

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