FOURTH-ORDER CONVERGENT TWO-STEP ITERATIVE ALGORITHM FOR SOLVING NON-LINEAR EQUATIONS

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ABSTRACT

In this paper we present fourth-order convergent two-step iterative algorithm for solving non-linear equations. This algorithm is refinement of the existing iterative algorithms. Numerical experiment shows this fact.

Keywords: *Two-step iterative algorithms, Convergence-Order, Numerical examples.*

INTRODUCTION

Finding the roots of a single variable non-linear equations efficiently, is a very basic and old problem in numerical analysis and has many applications in applied sciences.

We consider equation of the form

$$f(x) = 0 \tag{1.1}$$

where f is single variable non-linear function. The root-finding problem of such an equation, has always been very attractive area for the researchers, many methods for solving non-linear equations have been developed using error analysis techniques, quadrature rules or other techniques, see $[1-7]$. Recently, Mamta et al $[7]$, Kanwar et al. $[6]$, J. Chen and W. Li $[2]$ have suggested some iterative methods for solving non-linear equations.

Motivated by the research going on in this direction, we propose and analyze two-step iterative algorithms by combining J. Chen and W. Li $[2]$ and Mamta $[7]$ methods for solving non-linear equations. Numerical comparison between the algorithms $[2]$ and the newly developed algorithms is given. Numerical comparison shows that the new fourth-order two-step iterative algorithm is comparable with the existing algorithms and in many cases its performance is better. Our results can be considered as an improvement and refinement of the previously known results in the literature.

TWO-STEP ITERATIVE ALGORITHM

Here, we suggest new two-step iterative algorithm for solving non-linear equations.

Algorithm:

Step 1: *For initial guess* x_o, *a tolerance* $\varepsilon > 0$, *and for iterations* n, *set* $k = 0$.

if

$$f'(x_k) > 0,$$

then

$p_k = 1,$

else

$p_k = -1.$

Step 2: *Calculate*

$$z_k = x_k - \frac{2f(x_k)}{f'(x_k) + p_k\sqrt{f'^2(x_k) + 4f^2(x_k)}},$$

$$x_{k+1} = z_k \exp\left(-\frac{f(z_k)}{z_k f'(z_k)}\right).$$

Step 3: *If* $|x_{k+1} - x_k| < \varepsilon$ *or* $k > n$ *then stop.*

Step 4: *Set* $k \rightarrow k+1$ *and go to* ***Step 2.***

CONVERGENCE ANALYSIS

In this section, we discuss the convergence of Algorithm 1 .

Theorem *Let* $r \in [a,b]$ *be the simple root of differentiable function* $f:[a,b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ *for an open interval* (a,b). *If* x_o *is sufficiently close to* r *, then the iterative algorithm defined by Algorithm 1 has fourth-order convergence.*

Proof *The algorithm is given by*

$$z_n = x_n - \frac{2f(x_n)}{f'(x_n) \pm \sqrt{f'^2(x_n) + 4f^2(x_n)}},$$

and

$$x_{n+1} = z_n \exp\left(-\frac{f(z_n)}{z_n f'(z_n)}\right).$$

From (3.1), *we have*

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)\left(1 + \frac{f^2(x_n)}{f'^2(x_n)}\right)}.$$

Let r *be the exact root of* (1.1) , e_n *be the error at* n *th iteration, then*

$$e_n = x_n - r.$$

By the Taylor expansion about r, *we have*

$$\begin{aligned} f(x_n) &= f(r) + (x_n - r)f'(r) + \frac{1}{2!}(x_n - r)^2 f''(r) + \frac{1}{3!}(x_n - r)^3 f''(r) \\ &\quad + \frac{1}{4!}(x_n - r)^4 f^{(4)}(r) + \dots \\ &= e_n f'(r) + \frac{1}{2!}e_n^2 f''(r) + \frac{1}{3!}e_n^3 f''(r) + \frac{1}{4!}e_n^4 f^{(4)}(r) + O(e_n^5) \\ &= f'(r)\left(e_n + \frac{1}{2!}\frac{f''(r)}{f'(r)}e_n^2 + \frac{1}{3!}\frac{f'''(r)}{f'(r)}e_n^3 + \frac{1}{4!}\frac{f^{(4)}(r)}{f'(r)}e_n^4\right) + O(e_n^5) \\ &= f'(r)\left(e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4\right) + O(e_n^5), \end{aligned}$$

where

$$c_n = \frac{1}{n!}\frac{f^{(n)}(r)}{f'(r)}.$$

Similarly,

$$f'(x_n) = f'(r)(1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4) + O(e_n^5).$$

Dividing (3.4) *and* (3.5), *we get*

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2 c_3 - 4c_2^3 - 3c_4)e_n^4 + O(e_n^5).$$

From (3.6), *we get*

$$\frac{f^2(x_n)}{f'^2(x_n)} = e_n^2 - 2c_2 e_n^3 + (5c_2^2 - 4c_3)e_n^4 + O(e_n^5).$$

Now

$$\left[1 + \left(\frac{f(x_n)}{f'(x_n)}\right)^2\right]^{-1} = 1 - e_n^2 + 2c_2 e_n^3 - (5c_2^2 - 4c_3 - 1)e_n^4 + O(e_n^5).$$

From $(3.6 - 3.8)$ *and* (3.4), *we have:*

$$\begin{aligned} z_n &= x_n - e_n - c_2 e_n^2 + (2c_2^2 - 2c_3 - 1)e_n^3 + (7c_2 c_3 - 4c_2^3 + 3c_2 - 3c_4)e_n^4 \\ &= r + c_2 e_n^2 - (2c_2^2 - 2c_3 - 1)e_n^3 - (7c_2 c_3 - 4c_2^3 + 3c_2 - 3c_4)e_n^4. \end{aligned}$$

Expanding $f(z_n)$ *and* $f'(z_n)$ *about* r *and using* (3.9), *we get*

$$\begin{aligned} f(z_n) &= f(r) + (z_n - r)f'(r) + \frac{1}{2!}(z_n - r)^2 f''(r) + \dots \\ &= f'(r)\Big[c_2 e_n^2 - (2c_2^2 - 2c_3 - 1)e_n^3 \\ &\qquad - (7c_2 c_3 - 5c_2^3 + 3c_2 - 3c_4)e_n^4\Big], \end{aligned}$$

and

$$\begin{aligned}
f'(z_n) &= f'(r) + (z_n - r)f''(r) + \frac{1}{2!}(z_n - r)^2 f''(r) + \ldots \\
&= f'(r) + \left[c_2 e_n^2 - (2c_2^2 - 2c_3 - 1)e_n^3 \right. \\
&\quad \left. - (7c_2c_3 - 4c_2^3 + 3c_2 - 3c_4)e_n^4 \right] f''(r) \\
&\quad + \frac{1}{2!}\left(c_2 e_n^2\right)^2 f''(r) + \ldots \\
&= f'(r)\left[1 + 2c_2^2 e_n^2 - 2c_2(2c_2^2 - 2c_3 - 1)e_n^3 \right. \\
&\quad \left. - c_2(11c_2c_3 - 8c_2^3 + 6c_2 - 6c_4)e_n^4 \right].
\end{aligned}$$

From (3.11) *and* (3.12), *we have*

$$\begin{aligned}
&\frac{f(z_n)}{f'(z_n)} \\
&= \frac{c_2 e_n^2 - (2c_2^2 - 2c_3 - 1)e_n^3 - (7c_2c_3 - 5c_2^3 + 6c_2 - 3c_4)e_n^4}{1 + 2c_2^2 e_n^2 - 2c_2(2c_2^2 - 2c_3 - 1)e_n^3 - c_2(11c_2c_3 - 8c_2^3 + 6c_2 - 6c_4)e_n^4} \\
&= c_2 e_n^2 - (2c_2^2 - 2c_3 - 1)e_n^3 - (7c_2c_3 - 3c_2^3 + 6c_2 - 3c_4)e_n^4.
\end{aligned}$$

Taking the second order Taylor series expansion of (3.2), *we get*

$$\begin{aligned}
x_{n+1} &= z_n \left[1 - \frac{f(z_n)}{z_n f'(z_n)} + \left(\frac{f(z_n)}{z_n f'(z_n)} \right)^2 + \ldots \right] \\
&= z_n - \frac{f(z_n)}{f'(z_n)} + \frac{f^2(z_n)}{z_n f'^2(z_n)}.
\end{aligned}$$

Using (3.6) *and* (3.7) *in* (3.13) *, we get*

$$\begin{aligned}
x_{n+1} &= r + c_2 e_n^2 - (2c_2^2 - 2c_3 - 1)e_n^3 - (7c_2c_3 - 4c_2^3 + 3c_2 - 3c_4)e_n^4 \\
&\quad - c_2 e_n^2 + (2c_2^2 - 2c_3 - 1)e_n^3 + (7c_2c_3 - 3c_2^3 + 6c_2 - 3c_4 + r^{-1}c_2^2)e_n^4 \\
&\quad + O(e_n^5), \\
&= r + (c_2^3 - 3c_2 + r^{-1}c_2^2)e_n^4 + O(e_n^5), \\
e_{n+1} &= (c_2^3 - 3c_2 + r^{-1}c_2^2)e_n^4 + O(e_n^5),
\end{aligned}$$

which shows that the algorithm has fourth order convergence.

NUMERICAL EXAMPLES

Here, we perform some numerical tests to illustrate the performance of the new algorithm. We call our algorithm as **MCL (Mamta-Chen-Li algorithm)** and compare it with the Newton method (**NM**) and **Chen-Li** methods (formula- 9 and formula- 11) $[2]$, we name these methods as CL (9) and CL (11) . We consider here $\varepsilon = 1.0E-15$ as given tolerance and $n = 1000$, the maximum number of iterations to be performed. Numerical results are provided in the Table **4.1**.

Example	x_0	**CL** (9)	**CL** (11)	**NM**	**MCL**	$\|f(x_n)\|$
$\log(x) = 0$	5	9	4	D	2	$0.00000e+00$
$x - e^{\sin(x)} + 1 = 0$	4	8	6	$*$	5	$4.4409e-16$
$\tan^{-1}x + \sin x + x - 2 = 0$	4	8	6	10	5	$0.00000e+00$
$x^3 - 2x - 5 = 0$	4	8	7	7	5	$8.88178e-16$
$5x^3 - xe^x - 1 = 0$	3.8	10	5	22	7	$2.22045e-16$
$1 - 2\sin x = 0$	$\frac{\pi}{3}$	4	4	5	2	$0.00000e+00$
$(10-x)e^{-10x} - x^{10} + 1 = 0$	8	101	26	25	15	$9.99200e-16$
$-\frac{x^3+x-11}{3x^4-2x^2+5} = 0$	2.2	21	21	21	4	$6.8402e-17$
$(x-1)e^{-x} = 0$	1.5	5	5	7	4	$0.00000e+00$

In above Table **4.1**.

"D" stands for divergent.

"*" stands for not convergent.

CONCLUSION

We have proposed new two-step algorithm of convergence order four for solving single variable non-linear equation. On the basis of numerical tests, it follows that our algorithm is comparable with the existing two-step algorithms and in most of the cases, its performance is better than the algorithm mentioned in the table: 4.1. Our algorithm can be considered as an improvement and refinement of these algorithms.

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