SOME DERIVATIVE FREE ITERATIVE METHODS FOR SOLVING NONLINEAR EQUATIONS

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ABSTRACT

In this paper we present two new derivative free iterative methods for finding the zeros of the nonlinear equation f(x) = 0. Finding the zeros of the nonlinear equations is a classical problem in numerical analysis arises frequently in various branches of science and engineering. The aim of this paper is to develop some efficient methods to find the approximation of the root ω of the nonlinear equation f(x) = 0, without the evaluation of the derivatives. The new methods based on the central-difference and forward-difference approximations to derivatives. It is proved that one of the methods has cubic convergence and other method has fourth-order convergence. The benefit of these methods is that these methods do not need to calculate any derivative. Several examples illustrate that the convergence and efficiency of the new methods are better than previous methods.

Keywords: nonlinear equations, iterative methods, Newton's method, Central approximation, Derivative free method.

INTRODUCTION

In recent years many researchers have developed several iterative methods for solving nonlinear equations. In this paper we are going to develop efficient methods to find approximations of the root w of $f(x)=0$, without evaluation of derivatives.

A number of ways are considered by many researchers to improve the local order convergence of Newton's method by the expense of additional evaluations of the functions, derivatives and changes in the points of iterations (Frontini, M. & Sormani, E., 2003), (Homeier, H.H.H., 2003), (Homeier, H.H.H., 2005), (Kou et al., 2006), (Ozban, A.Y., 2004), (Weerakoom, S. & Fernando, T.G.I., 2000). There are several different methods in literature for the computation of the root w of the nonlinear equation $f(x)=0$. The most famous of these methods is the classical Newton's method (NM):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

which is a well-known basic method, converges quadratically in the neighbor-hood of simple root w. This method is not applicable when the derivative of any function is not defined in any interval. Therefore the Newton's method (1) was modified by Steffensen who replaced the first derivative $f'(x)$ in Newton's method by forward difference approximation

$$f'(x_n) \approx \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)},$$

and obtained the famous Steffensen's method (SM), (D. Kincaid and W. Cheney, 1996):

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}. \tag{2}$$

Both the above methods (1) and (2) are of quadratic convergence and require two functional evaluations per iteration but the best property of the Steffensen's method is that it is free from any derivative of the function, because some times the applications of the iteration methods which depend upon derivatives are restricted in engineering.

A steffensen-secant method was proposed by (Jain, 2007) free from derivatives (JM), which uses three functional evaluations per step and has cubic convergence. A family of Steffensen like methods (ZM) was derived by (Zheng et al., 2009), by applying forward difference approximation to the Weerakoon and Fernando method (Weerakoon and Fernando, 2000), which has cubic convergence and uses four functional evaluations per iteration. By applying central-difference and forward difference approximations (Dehghan and Hajarian, 2010), proposed four iterative methods free from derivatives, three of which have cubic and one has quadratic convergence. However two of these methods with cubic convergence require four functional evaluations per step.

In similar way (Codero et al., 2010) used central difference approximation

$$f'(x_n) \approx \frac{f(x_n + f(x_n)) - f(x_n - f(x_n))}{2f(x_n)},$$

to approximate the derivative in the fourth-order Ostrowski's method (Ostrowski, 1960) and obtained a new method (CM), free from derivatives which has fourth order convergence and requires four functional evaluations per iteration.

DESCRIPTION OF THE METHODS

We propose the following method (NM1) free from derivatives with fourth order convergence:

$$y_n = x_n - \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))}, \tag{3}$$

$$x_{n+1} = y_n - \frac{2f(y_n)f(x_n)}{g(u_n)[f(x_n + f(x_n)) - f(x_n - f(x_n))]}, \tag{4}$$

where $$u_n = \frac{f(y_n)}{f(x_n)}, \tag{5}$$

and $g(\lambda)$ is a real valued function to be determined later.

And the method (NM2) free from derivative with third order convergence:

$$y_n = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}, \tag{6}$$

$$x_{n+1} = y_n - \frac{f(y_n)f(x_n)}{g(u_n)[f(x_n + f(x_n)) - f(x_n)]}, \tag{7}$$

where $$u_n = \frac{f(y_n)}{f(x_n)}, \tag{8}$$

and $g(\lambda)$ is a real valued function to be determined later.

It can be seen that the method (3),(4) and (5) requires four functions evaluations per iteration with efficiency index 1.414 and the method (6),(7) and (8) comprises only three functional evaluations at each step with efficiency index 1.442. In the next section, a detailed description of the convergence analysis of the proposed methods has been given.

ANALYSIS OF CONVERGENCE

In this section we will present the analysis of convergence by giving mathematical proof for the order of convergence of the methods defined by (3), (4), (5) and (6), (7), (8).

Theorem 3.1

Let f: $\boldsymbol{D} \subseteq \boldsymbol{R} \to \boldsymbol{R}$ be a sufficiently differentiable function and $w \in \boldsymbol{D}$ be a simple zero of f in an open interval $\boldsymbol{D}$. Let g any function with $g(0) = 1$, $g'(0) = -2$ and $g''(0) < 1$. If x_0 is sufficiently close to w, then the method free from derivatives defined by (3),(4) and (5) has order of convergence 4 and satisfies the following error equation:

$$e_{n+1} = Ke_n^{\,4} + O(e_n^{\,5}) \tag{9}$$

where

$$K = \frac{c_2^3}{c_1^3}\left(\frac{g''(0)}{2} + 1\right) - c_2 c_3\left(1 + \frac{1}{c_1^2}\right)$$

and

$$e_n = x_n - w, \quad c_k = \frac{f^k}{k!}, k = 1,2,....$$

Proof.

The Taylor's expansion of $f(x_n)$ about w is

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^4 + O(e_n^5). \tag{10}$$

Furthermore, we have

$$f(x_n)^2 = c_1^2 e_n^2 + 2c_1 c_2 e_n^3 + (2c_3 c_1 + c_2^2) e_n^4 + O(e_n^5). \tag{11}$$

Again by using Taylor's expansion we can get

$$\begin{aligned} f(x_n + f(x_n)) = {} & c_1(1 + c_1)e_n + (3c_2 c_1 + c_2 + c_1^2 c_2)e_n^2 \\ & + (4c_3 c_1 + 2c_2^2 + 2c_2^2 c_1 + c_3 + 3c_3 c_1^2 + c_3 c_1^3)e_n^3 \\ & + (5c_1 c_4 + c_4 + 6c_4 c_1^2 + 4c_4 c_1^3 + c_4 c_1^4 + 5c_2 c_3 \\ & + 8c_2 c_3 c_1 + c_2^3 + 3c_2 c_3 c_1^2)e_n^4 + O(e_n^5) \end{aligned} \tag{12}$$

and

$$\begin{aligned} f(x_n - f(x_n)) = {} & -c_1(-1 + c_1)e_n + (-3c_2 c_1 + c_2 + c_1^2 c_2)e_n^2 \\ & + (-4c_3 c_1 - 2c_2^2 + 2c_2^2 c_1 + c_3 + 3c_3 c_1^2 - c_3 c_1^3)e_n^3 \\ & + (-5c_1 c_4 + c_4 + 6c_4 c_1^2 - 4c_4 c_1^3 + c_4 c_1^4 - 5c_2 c_3 \\ & + 8c_2 c_3 c_1 + c_2^3 - 3c_2 c_3 c_1^2)e_n^4 + O(e_n^5) \end{aligned} \tag{13}$$

which gives

$$f(x_n+f(x_n))-f(x_n-f(x_n))=2c_1^{\ 2}e_n+6c_2c_1e_n^{\ 2}+(4c_2^{\ 2}+8c_3c_1+2c_3c_1^{\ 3})e_n^{\ 3}+(6c_2c_3c_1^{\ 2}+10c_2c_3+8c_4c_1^{\ 3}+10c_1c_4)e_n^{\ 4}. \quad (14)$$

Now by substituting (11) and (14) in (3), we have

$$y_n=\frac{c_2}{c_1}e_n^{\ 2}+(-\frac{2c_2^{\ 2}}{c_1^{\ 2}}+2\frac{c_3}{c_1}+c_3c_1)e_n^{\ 3}+(-\frac{7c_2c_3}{c_1^{\ 2}}+3\frac{c_4}{c_1}+4c_1c_4+4\frac{c_2^{\ 3}}{c_1^{\ 3}}-c_2c_3)e_n^{\ 4}+O(e_n^{\ 5}). \quad (15)$$

The Taylor's series expansion of $f(y_n)$ can be written as

$$f(y_n)=c_2e_n^{\ 2}+(-2c_2^{\ 2}+c_3c_1^{\ 3}+2c_3c_1)\frac{e_n^{\ 3}}{c_1}+(-c_2c_3c_1^{\ 3}+4c_4c_1^{\ 4}+3c_4c_1^{\ 2}+5c_2^{\ 3}-7c_2c_3c_1)\frac{e_n^{\ 4}}{c_1^{\ 2}}+O(e_{n5}). \quad (16)$$

From (19) and (13) we obtain

$$u_n=\frac{f(y_n)}{f(x_n)}=\frac{c_2}{c_1}e_n+(c_1c_3-3\frac{c_2^{\ 2}}{c12}+2\frac{c_3}{c_1})e_n^{\ 2}+(4c_1c^4+3\frac{c_4}{c_1}-10\frac{c_2c_3}{c_1^{\ 2}}+8\frac{c_2^{\ 3}}{c_1^{\ 3}}-2c_2c_3)e_n^{\ 3}+(-8\frac{c_2^{\ 4}}{c_1^{\ 4}}-4c_2c_4-2\frac{c_3^{\ 2}}{c_1^{\ 2}}-c_3^{\ 2}+13\frac{c_2^{\ 2}c_3}{c_1^{\ 3}}+2\frac{c_3c_2^{\ 2}}{c_1})e_n^{\ 4}. \quad (17)$$

From the assumption of g and (17) we have

$$g(u_n)=g(0)+g'(0)u_n+g''(0)u_n^{\ 2}+O(u_{n3})=1-2\frac{c_2}{c_1}e_n+[2c_3(-c_1-\frac{2}{c_1})+2\frac{c_2^{\ 2}}{c_1^{\ 2}}(3+\frac{g''(0)}{4})]e_n^{\ 2}+\ldots \quad (18)$$

We can easily obtain from (14) and (18)

$$g(u_n)[f(x_n+f(x_n))-f(x_n-f(x_n))]=2c_1^{\ 2}e_n+2c_2c_1e_n^{\ 2}+(4c_2^{\ 2}-2c_3c_1^{\ 3}+c_2^{\ 2}g''(0))e_n^{\ 3}+(8c_4c_1^{\ 3}+3\frac{c_2^{\ 3}}{c_1}g''(0))+28\frac{c_2^{\ 3}}{c_1}-10c_2c_3c_1^{\ 2}+10c_1c_4-30c_2c_3)e_n^{\ 4}+O(e_n^{\ 5}). \quad (19)$$

From (10) and (16) it is not difficult to obtain

$$f(y_n)f(x_n) = c_1c_2e_n{}^3 + (-c_2{}^2 + c_3c_1{}^3 + 2c_3c_1)e_n{}^4 + O(e_n{}^5). \qquad (20)$$

Now by using (15), (19) and (20) in (4) we get

$$e_{n+1} = [\frac{c_2^3}{c_1^3}(\frac{g''(0)}{2}+1) - c_2c_3(1+\frac{1}{c_1^2})]e_n{}^4 + O(e_n{}^5). \qquad (21)$$

$$e_{n+1} = Ke_n{}^4 + O(e_n{}^5). \qquad (22)$$

where

$$K = \frac{c_2^3}{c_1^3}(\frac{g''(0)}{2}+1) - c_2c_3(1+\frac{1}{c_1^2})$$

Which shows that the new method (3), (4) and (5) is derivative free fourthly convergent method.

The next theorem shows that the method (6), (7) and (8) has cubic convergence.

Theorem 3.2

Under the same assumptions of theorem 3.1, the method free from derivatives defined by (6), (7) and (8) has order of convergence 3 and satisfies the following error equation:

$$e_{n+1} = Ke_n{}^3 + O(e_n{}^4) \qquad (23)$$

where

$$K = (-c_2{}^2 - \frac{c_2{}^2}{c_1})$$

and

$$e_n = x_n - w, \qquad c_k = \frac{f^k}{k!}, k = 1,2,....$$

Proof.

By substituting (11) and (12) in (6) and after some simple calculations, we obtain

$$\begin{aligned} y_n &= (c_2 + \frac{c_2}{c_1})e_n{}^2 + (2\frac{c_3}{c_1} + c_3c_1 - c_2{}^2 + 3c_3 - 2\frac{c_2{}^2}{c_1{}^2} - 2\frac{c_2{}^2}{c_1})e_n{}^3 \\ &+ (5\frac{c_2{}^3}{c_1{}^2} + 3\frac{c_2{}^3}{c_1} + 3\frac{c_4}{c_1} - 7c_2c_3 + 6c_4 + c_2{}^3 + c_4c_1{}^2 + 4\frac{c_2{}^3}{c_1{}^3} + 4c_1c_4 \\ &- 10\frac{c_2c_3}{c_1} - 2c_2c_3c_1 - 7\frac{c_2c_3}{c_1{}^2})e_n{}^4 + O(e_n{}^5). \end{aligned} \qquad (24)$$

By using Taylor's theorem, we have

$$f(y_n)=c_2(1+c_1)e_n^{\ 2}+(2_c^{\ 3}c_1+c_3c_1^{\ 3}-c_2^{\ 2}c_1^{\ 2}+3c_3c_1^{\ 2}-2c_2^{\ 2}$$
$$-2c_2^{\ 2}c_1)\frac{e_n^{\ 3}}{c_1}+(7c_1c_2^{\ 3}+4c_2^{\ 3}c_1^{\ 2}+3c_4c_1^{\ 2}-7c_2c_3c_1^{\ 3}$$
$$+6c_4c_1^{\ 3}+c_2^{\ 3}c_1^{\ 3}+c_4c_1^{\ 5}+5c_2^{\ 3}+4c_4c_1^{\ 4}-10c_2c_3c_1^{\ 2} \tag{25}$$
$$-2c_2c_3c_1^{\ 4}-7c_2c_3c_1)\frac{e_n^{\ 4}}{c_1^{\ 2}}.$$

From (25) and (10) we obtain

$$u_n=\frac{f(y_n)}{f(x_n)}=(c_2+\frac{c_2}{c_1})e_n+(2\frac{c_3}{c_1}-3\frac{c_2^{\ 2}}{c_1}+c_3c_1-3\frac{c_2^{\ 2}}{c_1^{\ 2}}-c_2^{\ 2}$$
$$+3c_3)e_n^{\ 2}+(c_2^{\ 3}-4c_2c_3-10\frac{c_2c_3}{c_1^{\ 2}}+5\frac{c_2^{\ 3}}{c_1}-2c_2c_3c_1+8\frac{c_2^{\ 2}}{c_1^{\ 3}}$$
$$+3\frac{c_4}{c_1}+4c_1c_4-14\frac{c_2c_3}{c_1}+6c_4+10\frac{c_2^{\ 3}}{c_1^{\ 2}}+c_4c_1^{\ 2})e_n^{\ 3}+(10c_5$$
$$-9c_2c_1^{\ 4}-6c_3 2c_1+60\frac{c_2^{\ 3}c_3}{c_1^{\ 2}}-25\frac{c_2c_4}{c_1}-\frac{c_5}{c_1^{\ 2}}-14\frac{c_2c_4}{c_1^{\ 2}}-c_3^{\ 2}c_1^{\ 2} \tag{26}$$
$$+37\frac{c_2^{\ 2}c_3}{c_1^{\ 3}}-2c_2c_4c_1^{\ 2}+3c_3c_1c_2^{\ 2}-c_2^{\ 4}-13c_3^{\ 2}-20c_2c^4+44\frac{c_3c_2^{\ 2}}{c_1}$$
$$-\frac{c_5}{c_1^{\ 3}}-15\frac{c_3^{\ 2}}{c_1}+\frac{c_5}{c_1^{\ 2}}-8\frac{c_3^{\ 2}}{c_1^{\ 2}}+5\frac{c_5}{c_1}-20\frac{c_2^{\ 4}}{c_1^{\ 2}}-30\frac{c_2^{\ 4}}{c_1^{\ 3}}-20\frac{c_2^{\ 4}}{c_1^{\ 4}}$$
$$-7\frac{c_2^{\ 4}}{c_1})e_n^{\ 4}+O(e_n^{\ 5}).$$

From the assumption on g and (26) we have

$$g(u_n)=g(0)+g'(0)u_n+g''(0)u_n^{\ 2}+O(u_n^{\ 3})$$
$$=1-2(\frac{c_2}{c_1}+2c_2)e_n+[-3c_3+6\frac{c_2^{\ 2}}{c_1}+2c_2^{\ 2}+6\frac{c_2^{\ 2}}{c_1^{\ 2}}-2c_3c_1 \tag{27}$$
$$-4\frac{c_3}{c_1}+\frac{g''(0)}{4}(2\frac{c_2}{c_1}+2c_2^{\ 2})]e_n^{\ 2}+O(e_n^{\ 3}).$$

It is not difficult to obtain

$$-3c_3c_1^{\ 2}+c_2^{\ 2}c_1^{\ 2}g''(0)-c_3c_1^{\ 3}+c_2^{\ 2}g''(0))e_n^{\ 3}$$
$$+(8c_1c_2^{\ 3}-15c_2c_3-28c_1c_2c_3-4c_1^{\ 3}c_2c_3+2c_1^{\ 2}c_2^{\ 3}$$
$$+6c_4c_1^{\ 2}-17c_1^{\ 2}c_2c_3+17c_2^{\ 3}+c_1^{\ 2}c_2^{\ 3}g''(0)+4c_4c_1^{\ 3}$$
$$+5c_1c_4+3\frac{c_2^{\ 3}}{c_1}g''(0)+4c_4c_1^{\ 3}+17c_2^{\ 3}+c_1^{\ 2}c_2^{\ 3}g''(0) \tag{28}$$
$$+4c_4c_1^{\ 3}+5c_1c_4+3\frac{c_2^{\ 3}}{c_1}g''(0)+5c_1c_2^{\ 3}g''(0)+14\frac{c_2^{\ 3}}{c_1}$$
$$+7c_2^{\ 3}g''(0)+c_4c_1^{\ 4})e_n^{\ 4}+O(e_n^{\ 5}).$$

From (7) and (22) we obtain

$$f(y_n)f(x_n) = c_1c_2(1+c_1)e_n^3 + (c_1^3c_3 - c_1c_2^2c_1^2c_2^2 - c_2^2 + 2c_1c_3 + 3c_1^2c_3)e_n^4 + O(e_n^5). \quad (29)$$

Now by using (24), (28), (29) in (7) we get

$$e_{n+1} = (-c_2^2 - \frac{c_2^2}{c_1})e_n^3 + O(e_n^4). \quad (30)$$

So,

$$e_{n+1} = Ke_n^3 + O(e_n^4). \quad (31)$$

Which shows that the new method (6), (7), (8) is a derivative free cubically convergent method.

NUMERICAL EXAMPLES

In this section we present some numerical examples by employing Nusrat Moin method (NM1) and Nusrat Moin method (NM2) to solve some nonlinear equations and compare it with Newton's method (NM), Steffensen's method (SM), (D. Kincaid and W. Cheney, 1996), Jain method (JM), (Jain, 2007), Zheng method (ZM), (Zheng et al., 2009), and Codero method (CM), (Codero et al., 2010). All computations are carried out with double arithmetic precision on Mapal 7. There are number of iterations and the number of functional evaluations in Table 1, required such that $|f(x_n)| < 10^{-15}$. Numerical results show the new derivative free methods (NM1) and (NM2) can compete with Newton's method (NM) and Steffensen's method (SM) and converge faster than JM, ZM, DHM, (Dehghan and Hajarian, 2010), in many cases.

Table 1.

$f(x)$	x_0	Iterations							NOFE						
		NM	SM	JM	ZM	CM	NM1	NM2	NM	SM	JM	ZM	CM	NM1	NM2
f_1	0.5	4	4	3	3	2	2	3	8	8	9	12	8	8	9
f_2	1.85	6	6	6	6	4	4	6	12	12	18	24	16	16	12
f_3	3.5	6	6	4	4	3	3	4	12	12	12	16	12	12	12
f_4	1	5	5	4	4	3	3	4	10	10	12	16	12	12	12
f_5	2	4	4	3	3	2	2	3	8	8	9	12	8	8	9
f_6	1.5	4	4	3	3	2	3	3	8	8	9	12	8	8	9
f_7	3	6	6	4	4	3	3	4	12	12	12	16	12	12	12
f_8	-3	14	14	10	10	8	8	10	28	28	30	40	32	32	30

The following functions are used in Table 1.

$f_1 = cosx\text{-}x,$ $\quad w = 0.73908513321516067$
$f_2 = (x\text{-}1)^3\text{-}2,$ $\quad w = 2.2599210498948734$
$f_3 = (x\text{-}1)^2\text{-}1,$ $\quad w = 2$
$f_4 = x^3 + 4x^2\text{-}10,$ $\quad w = 1.3652300134140969$
$f_5 = sinx\text{-}\frac{1}{2}x,$ $\quad w = 1.8954942670339809$

$f_6 = sin^2(x)\text{-}x^2 + 1,$ $w = 1.4044916482153411$
$f_7 = x^2\text{-}e^x\text{-}3x + 2,$ $w = 0.25753028543986084$
$f_8 = xe^{x^2} - sin^2(x) + 3cos(x) + 5,$ $w = -1.207647827130919$

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